

EFFECTIVE EQUIDISTRIBUTION OF TWISTED HOROCYCLE FLOWS AND HOROCYCLE MAPS

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ABSTRACT. We prove bounds for twisted ergodic averages for horocycle flows of hyperbolic surfaces, both in the compact and in the non-compact finite area case. From these bounds we derive effective equidistribution results for horocycle maps. As an application of our main theorems in the compact case we further improve on a result of A. Venkatesh, recently already improved by J. Tanis and P. Vishe, on a sparse equidistribution problem for classical horocycle flows proposed by N. Shah and G. Margulis, and in the general non-compact, finite area case we prove bounds on Fourier coefficients of cusp forms which are off the best known bounds of A. Good only by a logarithmic term. Our approach is based on Sobolev estimates for solutions of the cohomological equation and on scaling of invariant distributions for twisted horocycle flows.

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1. INTRODUCTION

In this paper we prove bounds for *twisted ergodic averages* for horocycle flows of hyperbolic surfaces, both in the compact and in the non-compact finite area case. From these bounds we derive effective equidistribution results for horocycle maps. We also improve upon a result of A. Venkatesh [25] on a question of N. Shah [19] on a sparse equidistribution problem for the horocycle flow. Finally, from our estimates in the non-compact finite area case we derive bounds for the Fourier coefficients of automorphic forms which coincide with the best known bounds due to A. Good [12] up to logarithmic terms.

Our main results can be stated as follows. Let $\{h_t\}$ denote the stable horocycle flow on the unit tangent bundle M of a finite area hyperbolic surface S of constant curvature -1 . Let $\{a_t\}$ be the geodesic flow on M and let dist be the distance function on $M \times M$ determined by a metric on M for which the orbits of $\{a_t\}$ are geodesics (the choice of this metric will appear in Section 2).

Let x_0 be a fixed point in M . For any $x \in M$, let $d_M(x) := \mathrm{dist}(x, x_0)$. For $A \in [0, 1]$ and $Q > 0$, let us define subsets of “Diophantine points”

$$(1) \quad M_{A,Q} := \{x \in M : d_M(a_t(x)) \leq At + Q \text{ for all } t > 0\}.$$

When M is compact, there is some $Q > 0$ such that $M_{0,Q} = M$. More generally, $\cup_{Q>0} M_{1,Q} = M$, and by the logarithm law of geodesics, for any $A \in (0, 1]$, the set $\cup_{Q>0} M_{A,Q}$ has full Haar measure, see [22]. In fact, for almost all $x \in M$ we have

$$(2) \quad \limsup_{t \rightarrow +\infty} \frac{d_M(a_t(x))}{\log |t|} = \frac{1}{2}.$$

For all $s \in \mathbb{R}$, let $W^s(M)$ denote the Sobolev space of square-integrable functions on the unit tangent bundle M with respect to the normalized volume measure.

Theorem 1.1. *For every $s > 7$ and for every $(A, Q) \in [0, 1) \times \mathbb{R}^+$ there is a constant $C_{s,A,Q} := C_{s,A,Q}(M) > 0$ such that the following bounds hold: for every $\lambda \in \mathbb{R} \setminus \{0\}$, for every $(x, T) \in M \times \mathbb{R}^+$ such that x and $h_T(x) \in M_{A,Q}$ and $|\lambda T| \geq e$, for every*

zero-average function $f \in W^s(M)$, we have

$$(3) \quad \left| \int_0^T e^{i\lambda t} f \circ h_t(x) dt \right| \leq C_{s,A,Q} \|f\|_s \left(1 + \frac{|\lambda|^{\frac{2A}{1-A}}}{|\lambda|^{1/6}}\right) T^{5/6 + \frac{2A}{1-A}} \log^{1/2}(|\lambda T|).$$

By the logarithm law of geodesics, for all $\varepsilon > 0$ there exists a measurable function $C_{s,\varepsilon} : M \rightarrow \mathbb{R}^+$, finite almost everywhere, such that, if $|\lambda T| \geq e$,

$$(4) \quad \left| \int_0^T e^{i\lambda t} f \circ h_t(x) dt \right| \leq C_{s,\varepsilon}(x) C_{s,\varepsilon}(h_T(x)) \|f\|_s \times \left(1 + \frac{1}{|\lambda|^{1/6}}\right) T^{5/6} \log^{3/2+\varepsilon}(|\lambda T|).$$

Moreover, when M is compact or when M is non-compact but x belongs to a (closed) cuspidal horocycle of length $T \geq 1$ such that $\lambda T \in 2\pi\mathbb{Z}$, there exists a constant $C_s := C_s(M) > 0$ such that, if $|\lambda T| \geq e$,

$$(5) \quad \left| \int_0^T e^{i\lambda t} f \circ h_t(x) dt \right| \leq C_s \|f\|_s \left(1 + \frac{1}{|\lambda|^{1/6}}\right) T^{5/6} \log^{1/2}(|\lambda T|).$$

Bounds in the regime $|\lambda T| \leq e$ can immediately be derived by integration by parts from bounds on ergodic integrals of the horocycle flow, which are well-known (see for instance [3], [6], [21]). In fact,

$$\int_0^T e^{i\lambda t} f \circ h_t(x) dt = e^{i\lambda T} \int_0^T f \circ h_t(x) dt - i\lambda \int_0^T e^{i\lambda t} \int_0^t f \circ h_\tau(x) d\tau dt,$$

whenever $|\lambda T| \leq e$ we have the bound

$$\left| \int_0^T e^{i\lambda t} f \circ h_t(x) dt \right| \leq \left| \int_0^T f \circ h_t(x) dt \right| + \frac{e}{T} \int_0^T \left| \int_0^t f \circ h_\tau(x) d\tau \right| dt.$$

We remark that a different integration by parts, which this time exploits the cancellations given by the (fast) oscillations of the exponential function, implies that for $f \in C^1(M)$ in the regime $|\lambda T| \geq T^{1+\beta}$ we have, for all $x \in M$ and for all $T \geq 1$,

$$\left| \int_0^T e^{i\lambda t} f \circ h_t(x) dt \right| \leq \|f\|_{C^1(M)} \frac{2+T}{\lambda} \leq 3\|f\|_{C^1(M)} T^{1-\beta}.$$

Theorem 1.5 of [6] provides a precise asymptotics for large $T > 0$ of the ergodic integral $\int_0^T f \circ h_t(x) dt$ in terms of invariant distributions for the horocycle flow. A refinement of the asymptotics, in terms of finitely additive measures on horocycles, is developed in [2] with applications to limit probability distributions for horocycle flows. In particular, in the compact case the following result holds. Let $\mu_0 > 0$ denote the smallest non-negative eigenvalue of the Laplace-Beltrami operator Δ_S of the hyperbolic surface S and let

$$v_0 := \begin{cases} \sqrt{1-\mu_0} & \text{if } \mu_0 \leq 1; \\ 0 & \text{if } \mu_0 > 1. \end{cases}$$

For every $s > 3$ there exists a constant $C_s > 0$ such that for every zero-average function $f \in W^s(M)$, for all $(x, T) \in M \times \mathbb{R}^+$, we have (see also [3])

$$\left| \int_0^T f \circ h_t(x) dt \right| \leq C_s \|f\|_s \left(1 + T^{\frac{1+\nu_0}{2}} + T^{1/2} \log(e+T) \right).$$

The equidistribution of horocycle flows on surfaces of constant negative curvature was proved by H. Furstenberg [9] in the compact case and by Dani [5] in the non-compact finite area case. In the first case the horocycle flow is uniquely ergodic, while in second case all orbits equidistribute except for finitely many one-parameter families of closed (cuspidal) horocycles.

The effective equidistribution, that is, bounds on the speed of convergence of ergodic averages, for horocycle flows of hyperbolic surfaces (the case $\lambda = 0$ in Theorem 1.1) has been investigated thoroughly in the past decades, both for compact and non-compact, finite area surfaces [26], [17], [3] (in the general geometrically finite case), [13], [6], [20], [21], [2] (which proves results on limit distributions of ergodic integrals in the compact case). All these results indicate that in general the speed of convergence of ergodic averages of sufficiently smooth functions depend on the *spectral gap* of the Laplace-Beltrami operator of the surface. In fact, a rather complete asymptotics for ergodic averages of smooth functions was established in [6] and later refined in [2], where results on limit distributions of probability distributions given by ergodic integrals were derived.

A striking feature of our effective equidistribution result (Theorem 1.1) in the regime $\lambda T \geq e$ is its independence from the spectral properties of the Laplace-Beltrami operator (“spectral gap”). To the best of our knowledge this phenomenon was first conjectured by Venkatesh (the third author of this paper learned of this conjecture directly from A. Venkatesh in Spring 2012, the second author from P. Vishe in Spring 2014). Indeed, the first effective bounds on twisted ergodic integrals of horocycle flows proved by A. Venkatesh [25] were not uniform with respect to the spectral gap. Recently, in work developed in parallel with this paper, the third author and P. Vishe have refined Venkatesh method thereby proving a bound independent of the spectral gap [24]. Our method is completely different from Venkatesh’s approach in [25] (refined by the third author and P. Vishe in [24]), which is based on effective equidistribution [3], [6] and estimates on decay of correlations (see for instance [16]) for horocycle flows, and our bounds are somewhat better (for instance for λ small the bound of the third author and P. Vishe [24] for twisted integrals is of the form $\lambda^{-1/2} T^{8/9}$).

Recent results on the effective equidistribution of horocycle maps by Venkatesh [25] in the compact case and by P. Sarnak and A. Ubis [18] for the modular surface have been motivated by *sparse equidistribution* problems for the horocycle flow, that is, by the questions whether the horocycle flow still equidistributes when sampled along a polynomial sequence of times (of fractional degree larger than 1) or along the prime numbers. The first question, which appears in the work of N. Shah [19], asks whether the horocycle flow equidistributes along polynomial sequences of any degree. We recall that by a general pointwise ergodic theorem proved by J. Bourgain [1] the answer is affirmative for almost all points, along

time sequences given by polynomials of any degree with integer coefficients. The second question comes up in the work of P. Sarnak and A. Ubis [18] on the independence of the Möbius functions with respect to all sequences generated by the horocycle flows on the modular surface. Both questions have also been asked by G. Margulis for general unipotent flows [14].

In Venkatesh's work [25] the *effective equidistribution of horocycle maps* is derived from the effective equidistribution by a direct argument based on Fourier expansion of delta measures on the line. We retain in this paper the approach of the third author's thesis [23] which consists in combining effective equidistribution results for the twisted cohomological equation with the complete description of invariant distributions and the solution to the cohomological equation for horocycle maps. Our main result can be stated as follows.

Theorem 1.2. *For every $s > 14$ and $\varepsilon > 0$, and for every $(A, Q) \in [0, 1) \times \mathbb{R}^+$, there is a constant $C_{s,\varepsilon,A,Q} := C_{s,\varepsilon,A,Q}(M) > 0$ such that the following holds. For every $L > 0$, for every $(x, N) \in M \times \mathbb{N} \setminus \{0\}$ such that x and $h_{NL}(x) \in M_{A,Q}$ and for every $f \in W^s(M)$, we have*

$$(6) \quad \left| \sum_{k=0}^{N-1} f \circ h_{Lk}(x) - \frac{1}{L} \int_0^{NL} f \circ h_t(x) dt \right| \leq C_{s,\varepsilon,A,Q} \|f\|_s \times \left((1 + L^{1/6+\varepsilon})(NL)^{5/6+\frac{2A}{1-A}} \log^{1/2} N + \frac{1 + L^{6+A+\varepsilon}}{L} \right).$$

By the logarithm law for geodesics, for all $\varepsilon > 0$ there exists a measurable function $C_{s,\varepsilon} : M \rightarrow \mathbb{R}^+$ that is finite almost everywhere and satisfies

$$(7) \quad \left| \sum_{k=0}^{N-1} f \circ h_{Lk}(x) - \frac{1}{L} \int_0^{NL} f \circ h_t(x) dt \right| \leq C_{s,\varepsilon}(x) C_{s,\varepsilon}(h_{NL}(x)) \|f\|_s \times \left((1 + L^{1/6+\varepsilon})(NL)^{5/6} \log^{3/2+\varepsilon} N + \frac{1 + L^{6+\varepsilon}}{L} \right).$$

In addition, there exists a constant $C_{s,\varepsilon} > 0$ such that for all $(x, N) \in M \times \mathbb{N} \setminus \{0\}$, whenever $f \in W^s(M)$ is a coboundary for the time- L horocycle map h_L we have

$$(8) \quad \left| \sum_{k=0}^{N-1} f \circ h_{Lk}(x) \right| \leq C_{s,\varepsilon} \|f\|_s \frac{1 + L^{2+\varepsilon}}{L} (e^{d_M(h_{-\frac{L}{2}}(x))} + e^{d_M(h_{L(N-\frac{1}{2})}(x))}).$$

Finally, when M is compact there exists a constant $C_{s,\varepsilon} := C_{s,\varepsilon}(M) > 0$ such that for every $(x, N) \in M \times \mathbb{N} \setminus \{0\}$ and for every $f \in W^s(M)$ we have

$$(9) \quad \left| \sum_{k=0}^{N-1} f \circ h_{Lk}(x) - \frac{1}{L} \int_0^{NL} f \circ h_t(x) dt \right| \leq C_{s,\varepsilon} \|f\|_s \times \left((1 + L^{1/6+\varepsilon})(NL)^{5/6} \log^{1/2} N + \frac{1 + L^{6+\varepsilon}}{L} \right).$$

In our paper we also derive from the bounds of Theorem 1.2 on the ergodic sums of horocycle maps the following result on Shah's question.

Theorem 1.3. *Let M be compact. For all $0 \leq \delta < \delta_0 = 1/13$, for all $f \in C(M)$ and for all $x \in M$, we have*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f \circ h_{n^{1+\delta}}(x) = \int_M f d\text{vol}.$$

The above result improves upon the corresponding results by Venkatesh in [25] (who had an explicit $\delta_0 \in (0, 1/48)$ but no uniform estimate with respect to the “spectral gap”) and by the third author and Vishe [24] (who had $\delta_0 = 1/26$). Our argument essentially follows Venkatesh’s, so our improvement in the exponent derives from our better bounds for twisted ergodic integrals. The main idea is to approximate polynomial sequences with arithmetic progressions with a small error for sufficiently long times. By its very nature this approach cannot go beyond the threshold $\delta_0 = 1$, so in any event a complete answer to Shah’s question is very far out of reach of current methods.

The work of P. Sarnak and A. Ubis [18] focuses on the case of horocycle maps on the modular surface, for which they prove an effective version of Dani’s equidistribution theorem for unipotent maps. However, this is far from enough to control the distribution of the horocycle flow “at prime times”, as the equidistribution of one-parameter unipotents on the product space (that is, of the joinings of the horocycle flow) plays a crucial role. This work has provided a crucial motivation for us in developing our scaling approach to twisted ergodic integrals, an approach that we hope can be applied to more general problems of effective distributions for unipotent flows, including perhaps joinings of the horocycle flow. Their approach is based on approximation of orbits of horocycle flows by segments of orbits on cuspidal horocycles, whose equidistribution is derived from bounds on the Fourier coefficients of automorphic forms. This approach is therefore limited to non-compact surfaces of finite area and it seems to fall rather short of the optimal exponent in the effective Dani’s theorem.

In our paper the case of non-compact, finite area surfaces can be handled by the same method as the compact case, by taking into account the speed of escape of geodesic orbits into the cups. By writing Fourier coefficients of cusp forms as twisted ergodic integrals along cuspidal horocycles, we derive bounds which coincide up to a logarithmic factor with the best bounds available, due to A. Good [12], for general, possibly non-arithmetic, non co-compact lattices of $SL(2, \mathbb{R})$. This coincidence seems to indicate that our results are presumably rather hard to improve upon, at least as far as the exponent of the polynomial bound is concerned.

Corollary 1.4. *Let $\{a_n\} \subset \mathbb{C}$ denote the sequence of the Fourier coefficients of a holomorphic cusp form f of even integral weight- k for any non co-compact lattice $\Gamma \subset SL(2, \mathbb{R})$. There is a constant $C_f > 0$ such that for all $n \in \mathbb{N} \setminus \{0\}$ we have*

$$|a_n| \leq C_f n^{k/2-1/6} (1 + \log n)^{1/2}.$$

We recall that for the modular lattice, and more generally for congruence lattices, the Ramanujan-Petersson conjecture, proved by Deligne, states that the sharp bound $|a_n| \leq C_{f,\varepsilon} n^{k/2-1/2+\varepsilon}$ holds. To the authors best knowledge it is an open

question whether the optimal bound holds for general non co-compact lattices or even whether Good's bound can be improved.

We conclude the introduction with an outline of the methods of our paper. We recall that the main idea underlying all advances on the effective equidistribution of the horocycle flow is the fundamental fact that the orbit foliation of the horocycle flow is invariant under the geodesic flow, hence the horocycle flow is renormalized by the geodesic flow. In other terms, such results establish refined versions of the exponential decay of correlation for the geodesic flow. In our paper, we follow the approach first developed by the first two authors to prove effective ergodicity results for higher step nilflows [8]. We view the twisted ergodic integrals for the horocycle flow as special ergodic integrals for the product of the horocycle flow and of a linear flow on a circle. Our goal thus becomes to prove effective ergodicity results for the above product flow. Our proof of effective equidistribution is based on a scaling argument which is a generalization of the renormalization method developed in the work of the first two authors to prove effective equidistribution of horocycle flows [8]. It consists in an analysis, based on the theory of unitary representations for the group $\mathrm{SL}(2, \mathbb{R})$ of a scaling operator on the space of *invariant distributions* for the appropriate *cohomological equation*. In the present case the scaling is not induced by a renormalization dynamics. The exponent in our effective equidistribution theorem is the optimal scaling exponent of invariant distributions for the twisted horocycle flow. The logarithmic factor arises from the control of the geometry of the rescaled metric structure or, equivalently, from estimates related to close return times of the horocycle flow. The relevant measure of the degeneration of the geometry is a notion of injectivity radius, called the *average width* of a horocycle arc, already introduced in [8] for nilflows.

2. STATEMENT OF RESULTS

Let $\Gamma < \mathrm{SL}(2, \mathbb{R})$ be any lattice. The group $\mathrm{SL}(2, \mathbb{R})$ acts by right multiplication on the quotient manifold $M := \Gamma \backslash \mathrm{SL}(2, \mathbb{R})$. The Haar measure on $\mathrm{SL}(2, \mathbb{R})$ induces a right-invariant volume form vol on M which we normalize so that $\mathrm{vol}(M) = 1$. We recall that (X, U, V) is the basis of $\mathfrak{sl}_2(\mathbb{R})$ given by

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

These matrices satisfy the commutation relations

$$[X, U] = 2U, \quad [X, V] = -2V, \quad [U, V] = X;$$

The center of the enveloping algebra of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ is one-dimensional and is generated by the *Casimir operator*

$$(10) \quad \square = -X^2 - 2(UV + VU).$$

The Casimir operator commutes with the action of the full group $\mathrm{SL}(2, \mathbb{R})$ on the enveloping algebra, hence the differential operator induced by the Casimir operator on any unitary representation is $\mathrm{SL}(2, \mathbb{R})$ -invariant.

The flows on M defined, for all $x \in M$ and $t \in \mathbb{R}$, by the formulas

$$a_t(x) = x \exp(tX/2), \quad h_t(x) = x \exp(tU), \quad \bar{h}_t(x) = x \exp(tV),$$

are, by definition, the geodesic, stable and unstable horocycle flow, respectively. By “horocycle flow” we shall mean the stable horocycle flow $\{h_t\}$ generated by the vector field U on M . Our “twisted horocycle flows” will be product flows on $M \times \mathbb{T}$ of the horocycle flow on M with with a linear flow on the circle

$$\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}.$$

Let $L^2(M \times \mathbb{T})$ be the space of complex-valued, square-integrable functions on $M \times \mathbb{T}$. Let K denote the vector field on \mathbb{T} that is given by ordinary differentiation. The Laplace-Beltrami operator on $M \times \mathbb{T}$ is an elliptic second order operator. It is a non-negative essentially self-adjoint operator on $L^2(M \times \mathbb{T})$ and is denoted by

$$\Delta := -K^2 - X^2 - 2(U^2 + V^2).$$

For any $s > 0$, the operator $(I + \Delta)^{s/2}$ is defined by the spectral theorem. The Sobolev space $W^s(M \times \mathbb{T}) \subset L^2(M \times \mathbb{T})$ is defined to be the maximal domain of $(I + \Delta)^{s/2}$ on $M \times \mathbb{T}$ and is endowed with the inner product

$$\langle F, G \rangle_{W^s(M \times \mathbb{T})} := \langle (I + \Delta)^s F, G \rangle_{L^2(M \times \mathbb{T})}.$$

We denote the corresponding norm by

$$\|F\|_s := \langle F, F \rangle_{W^s(M \times \mathbb{T})}^{1/2}.$$

The distributional dual space to $W^s(M \times \mathbb{T})$ is denoted

$$W^{-s}(M \times \mathbb{T}) := (W^s(M \times \mathbb{T}))'$$

with norm denoted $\|\cdot\|_{-s}$. The space of smooth vectors in $L^2(M \times \mathbb{T})$ and its distributional dual space are denoted

$$(11) \quad \begin{aligned} W^\infty(M \times \mathbb{T}) &:= \cap_{s \geq 0} W^s(M \times \mathbb{T}), \text{ and} \\ W^{-\infty}(M \times \mathbb{T}) &:= \cup_{s \geq 0} W^{-s}(M \times \mathbb{T}), \end{aligned}$$

respectively. We remark that when M is compact, then $W^\infty(M \times \mathbb{T}) = C^\infty(M \times \mathbb{T})$ and $W^{-\infty}(M \times \mathbb{T}) = \mathcal{D}'(M \times \mathbb{T})$.

Let $W^s(M)$ be the subspace of functions in $W^s(M \times \mathbb{T})$ that are constant with respect to the natural circle action on $M \times \mathbb{T}$, which is endowed with the same inner product $\langle \cdot, \cdot \rangle_s$. Let $C^\infty(M)$ and $\mathcal{E}'(M)$ be defined as in (11).

Our results will often be proven using Sobolev norms involving only the K , X and V derivatives and the Casimir operator. Such *foliated Sobolev spaces* are defined as follows. The Casimir operator \square is an essentially self-adjoint operator on $L^2(M)$, hence on $L^2(M \times \mathbb{T})$. Also notice that the foliated Laplacian

$$\widehat{\Delta} := -K^2 - X^2 - V^2$$

is a non-negative essentially self-adjoint differential operator on $L^2(M \times \mathbb{T})$. For any $r, s \geq 0$, let $\widehat{W}^{r,s}(M \times \mathbb{T})$ be the Sobolev space that is the maximal domain of $(I + \square^2)^{r/2}(I + \square^2 + \widehat{\Delta}^2)^{s/2}$ on $L^2(M \times \mathbb{T})$ with inner product

$$\langle F, G \rangle_{\widehat{W}^{r,s}(M \times \mathbb{T})} := \langle (I + \square^2)^{r/2}(I + \square^2 + \widehat{\Delta}^2)^{s/2} F, G \rangle_{L^2(M \times \mathbb{T})}.$$

Let us define the norm on $\widehat{W}^{r,s}(M \times \mathbb{T})$ to be

$$|F|_{r,s} := \langle F, F \rangle_{\widehat{W}^{r,s}(M \times \mathbb{T})}^{1/2}.$$

The dual space of $\widehat{W}^{r,s}(M \times \mathbb{T})$ is denoted $\widehat{W}^{-r,-s}(M \times \mathbb{T})$ with norm $|\cdot|_{-r,-s}$.

2.1. Twisted horocycle flows. For any $\lambda \in \mathbb{R}^*$, the *twisted horocycle flow* is the flow $\{\phi_t^\lambda\}_{t \in \mathbb{R}}$ on $M \times \mathbb{T}$ generated by the vector field $(U + \lambda K)$. The main theorems of this paper concern the quantitative equidistribution of this flow, and its applications. As in [6], [7], [8] and [23], our analysis is based on invariant distributions and on bounds for solutions of a cohomological equation, and it is carried out in irreducible, unitary representation spaces.

Because the rate equidistribution of the horocycle flow has been completely understood in [6], we restrict our attention to $(U + \lambda K)$ -invariant distributions that are not U -invariant. We denote these spaces of infinite-order distributions by

$$\mathcal{I}_\lambda(\Gamma) := \{ \mathcal{D} \in \mathcal{E}'(M \times \mathbb{T}) : (U + \lambda K)\mathcal{D} = 0, \text{ and } U\mathcal{D} \neq 0 \}.$$

The space $\mathcal{I}_\lambda(\Gamma)$ is filtered by subspaces of invariant distributions of finite order. For all $r, s \geq 0$ we denote

$$\begin{aligned} \mathcal{I}_\lambda^s(\Gamma) &:= \{ \mathcal{D} \in W^{-s}(M \times \mathbb{T}) : (U + \lambda K)\mathcal{D} = 0, \text{ and } U\mathcal{D} \neq 0 \}; \\ \mathcal{I}_\lambda^{r,s}(\Gamma) &:= \{ \mathcal{D} \in \widehat{W}^{-r,-s}(M \times \mathbb{T}) : (U + \lambda K)\mathcal{D} = 0, \text{ and } U\mathcal{D} \neq 0 \}. \end{aligned}$$

If $\lambda \notin \mathbb{Z}$ and $\mathcal{D} \in \mathcal{I}_\lambda(\Gamma)$, then $U\mathcal{D} \neq 0$.

Theorem 2.1. *For all $\lambda \in \mathbb{R}^*$ the space $\mathcal{I}_\lambda(\Gamma)$ of invariant distribution for the twisted horocycle flow which are not horocycle flow invariant is an infinite dimensional subspace of the foliated Sobolev space $\widehat{W}^{0, -(1/2+)}(M \times \mathbb{T})$.*

For $f \in C^\infty(M \times \mathbb{T})$, invariant distributions appear in the study of solutions of the cohomological equation for the twisted horocycle flow:

$$(12) \quad (U + \lambda K)g = f.$$

Let

$$\text{Ann}_\lambda(\Gamma) := \{ f \in C^\infty(M \times \mathbb{T}) : \mathcal{D}(f) = 0 \text{ for all } \mathcal{D} \in \mathcal{I}_\lambda(\Gamma) \}.$$

Theorem 2.2. *For every function $f \in \text{Ann}_\lambda(\Gamma) \subset C^\infty(M \times \mathbb{T})$, there is a solution $g \in C^\infty(M \times \mathbb{T})$ of the twisted cohomological equation*

$$(U + \lambda K)g = f,$$

satisfying the following Sobolev estimates. For all $r, s \geq 0$, there is a constant $C_{r,s} := C_{r,s}(\Gamma) > 0$ such that with respect to the foliated Sobolev norms

$$|g|_{r,s} \leq \frac{C_{r,s}}{|\lambda|} (1 + |\lambda|^{-s}) |f|_{r+3s, s+1},$$

hence, for all $s \geq 0$ there is a constant $C_s := C_s(\Gamma) > 0$ such that with respect to the full Sobolev norms

$$\|g\|_s \leq \frac{C_s}{|\lambda|} (1 + |\lambda|^{-s}) \|f\|_{4s+1}.$$

From Theorem 2.1 and Theorem 2.2 together with a trace theorem and a quantitative analysis of the returns of the horocycle flow we will derive effective equidistribution results for the twisted horocycle flow $\{\phi_t^\lambda\}_{t \in \mathbb{R}}$ on $M \times \mathbb{T}$.

Let $\bar{x} \in M \times \mathbb{T}$ and $T \geq 1$. Let us consider the ergodic integral

$$(13) \quad \frac{1}{T} \int_0^T (\phi_t^\lambda(\bar{x}))^* dt,$$

as a distribution in $\widehat{W}^{0, -(1+)}(M \times \mathbb{T})$, whose regularity follows from a trace theorem discussed in Subsection 5.2.

On smooth functions which are constant with respect to the natural circle action on $M \times \mathbb{T}$ (on the second factor), the ergodic integral of the twisted horocycle flow restricts to the ergodic integral for the horocycle flow on M . The rate of equidistribution of the horocycle flow has been completely understood in [6], so we consider functions such that $\int_{\mathbb{T}} F = 0$, that is, functions which are in the orthogonal complement of the subspace of functions invariant under the above circle action.

We prove an effective equidistribution theorem for the twisted horocycle flow on finite volume manifolds M under some Diophantine conditions. In Section 6.1 we introduce a function $C_\Gamma : M \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ whose growth reflects the Diophantine properties of points. In particular, for every $A \in [0, 1)$ and every $Q > 0$ let $M_{A,Q} \subset M$ denote the set of Diophantine points introduced in formula (1). The function C_Γ will satisfy the following properties. There exists a constant $C_{\Gamma,A,Q} > 0$ such that for all $x \in M_{A,Q}$ and all $T \geq 1$,

$$C_\Gamma(x, T) \leq C_{\Gamma,A,Q} T^{\frac{2A}{1-A}}.$$

By the logarithm law of geodesics, for almost all $x \in M$, for all $\varepsilon > 0$ there exists a constant $C_\varepsilon(x) > 0$ such that for all $T \geq 1$,

$$(14) \quad C_\Gamma(x, T) \leq C_\varepsilon(x) (1 + \log^{1+\varepsilon} T).$$

In addition, if M is compact then there is a constant $C_\Gamma > 1$, depending only on Γ , such that then

$$C_\Gamma(x, T) \leq C_\Gamma.$$

Theorem 2.3. *For every $s > 2$ and $r \geq 5s - 3$, there is a constant $C_{r,s} := C_{r,s}(\Gamma) > 0$ such that the following holds. For any $\bar{x} = (x, \theta) \in M \times \mathbb{T}$, and for any $T \geq e$, there are distributions $\mathcal{D}_{\bar{x}, \lambda, T}^{r,s} \in \mathcal{J}_\lambda^{r,s}(\Gamma)$ and $\mathcal{R}_{\bar{x}, \lambda, T}^{r,s} \in \widehat{W}^{-r,-s}(M)^\perp \subset \widehat{W}^{-r,-s}(M \times \mathbb{T})$, orthogonal to $\mathcal{J}_\lambda^{r,s}(\Gamma)$ in $\widehat{W}^{-r,-s}(M)^\perp$, such that for any $F \in \widehat{W}^{r,s}(M \times \mathbb{T})$ satisfying $\int_{\mathbb{T}} F = 0$, we have*

$$(15) \quad \int_0^T F \circ \phi_t^\lambda(\bar{x}) dt = \mathcal{D}_{\bar{x}, \lambda, T}^{r,s}(F) T^{5/6} + \mathcal{R}_{\bar{x}, \lambda, T}^{r,s}(F),$$

where the following bounds hold:

$$(16) \quad \begin{aligned} |\mathcal{R}_{\bar{x}, \lambda, T}^{r,s}|_{-r,-s}^2 &\leq C_{r,s} \frac{1 + |\lambda|^{-2(s-1)}}{|\lambda|^2} [C_\Gamma(x, T) + C_\Gamma(h_T(x), T)]^2; \\ |\mathcal{D}_{\bar{x}, \lambda, T}^{r,s}|_{-r,-s}^2 &\leq C_{r,s} (1 + |\lambda|^{-8s}) [C_\Gamma(x, T) + C_\Gamma(h_T(x), T)]^2 \log T. \end{aligned}$$

Remark 2.4. *The above estimates are independent of the spectral gap of the Laplace-Beltrami operator on the underlying hyperbolic surface.*

In the regime $|\lambda T| \geq e$ and for $|\lambda| \leq e$ the above result can be improved by a scaling argument based on the action of the geodesic flow. In this case, for all $x \in M$ let us defined the constant $C_{\Gamma,\lambda}(x, T) > 0$ by the formula

$$(17) \quad C_{\Gamma,\lambda}(x, T) := C_{\Gamma}(a_{\log|\lambda|}^{-1}(x), |\lambda T|).$$

Corollary 2.5. *For every $s > 7$, there is a constant $C_s := C_s(\Gamma) > 0$ such that the following holds. For every $\lambda \in \mathbb{R} \setminus \{0\}$, the following bounds on the twisted ergodic integrals along the horocycle flow holds: for every $(x, T) \in M \times \mathbb{R}^+$ and for every zero-average function $f \in W^s(M)$, we have, if $|\lambda T| \geq e \geq |\lambda|$,*

$$(18) \quad \left| \int_0^T e^{i\lambda t} f \circ h_t(x) dt \right| \leq \frac{C_s}{|\lambda|} \|f\|_s \times [C_{\Gamma,\lambda}(x, T) + C_{\Gamma,\lambda}(h_T(x), T)] |\lambda T|^{5/6} \log^{1/2}(|\lambda T|).$$

Proof. For every $\lambda \in \mathbb{R}^*$, let $\{h_t^{U/|\lambda|}\}$ denote the horocycle flow with generator $U/|\lambda|$, which is a linear time-change of the stable horocycle flow $\{h_t\} = \{h_t^U\}$. By the change of variable formula, assuming that $|\lambda| \leq e$,

$$\int_0^T e^{i\lambda t} f \circ h_t(x) dt = \int_0^{|\lambda T|} e^{i\lambda t/|\lambda|} f \circ h_t^{U/|\lambda|}(x) \frac{dt}{|\lambda|}.$$

Let $a_{\log|\lambda|}$ be the geodesic map such that $h_t^{U/|\lambda|} = a_{\log|\lambda|} \circ h_t^U \circ a_{\log|\lambda|}^{-1}$. It follows from this formula that

$$\frac{d}{dt} f \circ h_t^{U/|\lambda|} \circ a_{\log|\lambda|} = \frac{d}{dt} f \circ a_{\log|\lambda|} \circ h_t^U.$$

Hence, we get the formula

$$|\lambda|^{-1} U f \circ a_{\log|\lambda|} = U(f \circ a_{\log|\lambda|}),$$

which implies that

$$|\lambda| V f \circ a_{\log|\lambda|} = V(f \circ a_{\log|\lambda|}).$$

Since our bounds are in terms of the foliated Sobolev norms, by the above choice of the geodesic map we have that

$$(19) \quad |f \circ a_{\log|\lambda|}|_{r,s} \leq \max\{|\lambda|^j |0 \leq j \leq s\} |f|_{r,s}.$$

Now, since $|\lambda| \leq e$, the above theorem yields a bound

$$(20) \quad \left| \int_0^{|\lambda T|} e^{i\lambda t/|\lambda|} f \circ a_{\log|\lambda|} \circ h_t^U \circ a_{\log|\lambda|}^{-1}(x) \frac{dt}{|\lambda|} \right| \leq \frac{C_{r,s}}{|\lambda|} |f|_{r,s} \times [C_{\Gamma,\lambda}(x, T) + C_{\Gamma,\lambda}(h_T(x), T)] (\lambda T)^{5/6} \log^{1/2}(|\lambda T|).$$

The argument is therefore complete. \square

Proof of Theorem 1.1. Let $A \in [0, 1)$ and $Q > 0$. By definition of $M_{A,Q}$, we have for all $x \in M_{A,Q}$, for all $\lambda \in \mathbb{R}^*$ and for all $t \geq 0$,

$$d(a_t(a_{\log|\lambda|}^{-1}(x))) \leq A(t - \log|\lambda|) + Q.$$

Then it follows from Remark 6.3 and Lemma 6.5 that there exists a constant $C_{\Gamma,A,Q} > 0$ such that whenever $x, h_T(x) \in M_{A,Q}$, if $|\lambda| \leq e$ we have

$$C_{\Gamma,\lambda}(h_T(x), T) + C_{\Gamma,\lambda}(x, T) \leq 2C_{\Gamma,A,Q}|\lambda T|^{\frac{2A}{1-A}};$$

by the logarithmic law of geodesics, for almost all $x \in M$ and for all $\varepsilon > 0$, there exists a constant $C_\varepsilon(x) > 0$ such that

$$C_{\Gamma,\lambda}(x, T) \leq C_\varepsilon(x)[1 + \log^{1+\varepsilon}(|\lambda T|)].$$

By the above bounds on the constants $C_{\Gamma,\lambda}(x, T)$ under the relevant Diophantine conditions, the statement of Theorem 1.1 is an immediate consequence of Theorem 2.3 for $|\lambda| \geq e$ and of Corollary 2.5 for $|\lambda| \leq e$.

Now assume that x lies on the cuspidal horocycle γ_T of length $T > 0$. We remark that, under the assumption that $\lambda T \in 2\pi\mathbb{Z}$, for all $s > 0$ and for all continuous functions f on M we have

$$\begin{aligned} \left| \int_0^T e^{i\lambda t} f \circ h_t(h_s(x)) dt \right| &= \left| \int_s^{T+s} e^{i\lambda(t-s)} f \circ h_t(x) dt \right| \\ &= \left| \int_s^{T+s} e^{i\lambda t} f \circ h_t(x) dt \right| = \left| \int_0^T e^{i\lambda t} f \circ h_t(x) dt \right|. \end{aligned}$$

In other terms in this case the modulus of a twisted horocycle integral along a cuspidal horocycle does not depend on the initial point.

Let $K_\Gamma > 0$ be a constant such that all cuspidal horocycles of unit length on M are contained in the compact set $\{x \in M : d_M(x) \leq K_\Gamma\}$. Let $C_\Gamma > 0$ denote the constant introduced below in Lemma 6.4.

By hyperbolic geometry for any given cusp and for any $x' \in M$, which does not belong to an unstable cuspidal horocycle for that cusp, there exists a (unique) point on every stable cuspidal horocycle such that the backward geodesic orbits of x and x' are asymptotic. Because there exists a dense set of points in M with relatively compact backward orbit, it follows that on every stable cuspidal horocycle there is a dense set of points with relatively compact *backward* geodesic orbit.

Thus, there exists a positive integer $n_\Gamma > 0$ such that any stable cuspidal horocycle γ_T of length $T > 0$ has a partition $\{x_1, \dots, x_{n_\Gamma}\}$ such that for all $k \in \{1, \dots, n_\Gamma\}$ (modulo n_Γ), there exists

$$T_k \in (0, \frac{C_\Gamma T}{10K_\Gamma})$$

such that $x_{k+1} = h_{T_k}(x_k)$ and x_k belongs to the relatively compact backward orbit of a point on the cuspidal horocycle of unit length. There exists therefore a constant $K'_\Gamma > 0$ such that the following bound holds:

$$\max_{0 \leq y \leq \log T} d_M(a_y(x_k)) \leq K'_\Gamma.$$

Since for all $T \geq 1$ the loop $a_{\log T}(\gamma_T)$ is a cuspidal horocycle of unit length, we have by the definition of K_Γ that for all $x \in \gamma_T$,

$$d_M(a_{\log T}(x)) \leq K_\Gamma.$$

hence, for each k , it follows from Lemma 6.5 and the condition $T_k < \frac{C_\Gamma T}{10K_\Gamma}$ that

$$(21) \quad C_\Gamma(x_k, T_k) = \max_{0 \leq t \leq T_k} c_\Gamma(x_k, t) \leq \left(\frac{10}{C_\Gamma}\right)^2 e^{2K'_\Gamma}.$$

Then we conclude from Theorem 2.3 and Corollary 2.5 that there exists a constant $C_s := C_s(\Gamma) > 0$ such that if $|\lambda T| \geq e$,

$$\left| \int_0^{T_k} e^{i\lambda t} f \circ h_t(x_k) dt \right| \leq C_s \|f\|_s \left(1 + \frac{1}{|\lambda|^{1/6}}\right) T^{5/6} \log^{1/2}(|\lambda T|).$$

Finally, the statement follows from finiteness of the partition. \square

2.2. Fourier coefficients of cusp forms.

Proof of Theorem 1.4. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be an arbitrary lattice containing the unipotent element $\begin{pmatrix} 1 & \\ 0 & 1 \end{pmatrix}$. Such lattices are not co-compact. Let $k \in \mathbb{N}$ be even, let f be a holomorphic cusp form of weight- k for Γ . Let f have the Fourier expansion

$$f = \sum_{n>0} a_n e^{2\pi i n z},$$

so the coefficients $\{a_n\}_{n>0} \subset \mathbb{C}$ are given by

$$a_n = e^{2\pi i} \int_{\mathbb{R}/\mathbb{Z}} f\left(t + \frac{i}{n}\right) e^{-2\pi i n t} dx.$$

Following Section 1.3.4 of [25], let \tilde{f} be the lift of f to $\Gamma \backslash \mathrm{SL}(2, \mathbb{R})$ given by

$$\tilde{f} : \Gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow f\left(\frac{ai+b}{ci+d}\right) (ci+d)^{-k}.$$

Then

$$(22) \quad \tilde{f} \in C^\infty(\Gamma \backslash \mathrm{SL}(2, \mathbb{R})).$$

Let $x_n = \Gamma \begin{pmatrix} n^{-1/2} & 0 \\ 0 & n^{1/2} \end{pmatrix} \in M$. Then

$$\tilde{f} \circ h_t(x_n) = n^{-k/2} f\left(\frac{i+t}{n}\right).$$

Consequently, we have as in formula (1.7) of [25] that

$$(23) \quad \begin{aligned} a_n &= e^{2\pi i} \int_{\mathbb{R}/\mathbb{Z}} f\left(t + \frac{i}{n}\right) e^{-2\pi i n t} dt \\ &= e^{2\pi i} n^{-1} \int_{\mathbb{R}/n\mathbb{Z}} f\left(\frac{i+t}{n}\right) e^{-2\pi i t} dt \\ &= e^{2\pi i} n^{k/2-1} \int_0^n \tilde{f} \circ h_t(x_n) e^{-2\pi i t} dt. \end{aligned}$$

The twisted integral (23) is over a closed horocycle of length n . Because \tilde{f} is smooth, we may take any $s > 7$, and Theorem 1.1 gives a constant $C_{r,s,f} > 0$ such that

$$|e^{2\pi n^{k/2-1}} \int_0^n \tilde{f} \circ h_t(x_n) e^{-2\pi i t} dt| \leq C_{r,s,f} n^{k/2-1/6} \log^{1/2}(e+n).$$

Theorem 1.4 is now immediate from (23). \square

2.3. Horocycle maps. A precise description of the space of invariant distributions and the statement of our effective equidistribution theorem for horocycle maps require that we recall of the theory of unitary representations of the group $\mathrm{SL}(2, \mathbb{R})$.

There are four classes of irreducible, unitary representations H_μ of $\mathrm{SL}(2, \mathbb{R})$. They are parameterized by the Casimir operator \square and termed the principal series, the complementary series, the discrete series and the mock discrete series. We have the following cases:

- When $\mu \in (0, 1)$, then H_μ is in the complementary series.
- When $\mu > 1$, then H_μ is in the principal series.
- When $\mu = 1$, then H_μ is in the mock discrete series or the principal series.
- When $\mu \leq 0$, then H_μ is in the discrete series.

The standard line and upper half-plane models for irreducible representations of $\mathrm{SL}(2, \mathbb{R})$ are discussed in Appendix A.

The irreducible, unitary representations of $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{T}$ are parameterized by tuples $(m, \mu) \in \mathbb{Z} \times \mathrm{spec}(\square)$, and they are denoted

$$H_{m,\mu} := H_\mu \otimes e_m,$$

where $e_m \in L^2(\mathbb{T})$ is given by

$$e_m(t) := e^{imt}.$$

We will refer to a representation $H_{m,\mu}$ as a principal series representation (resp. complementary series, discrete series, or mock discrete series) if H_μ is.

The regular representation $L^2(M \times \mathbb{T})$ of $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{T}$ decomposes as a direct sum or integral of irreducible, unitary representation spaces $H_{m,\mu}$, which occur with at most finite multiplicity. By irreducibility, vector fields are decomposable in the sense that, for any $s \in \mathbb{R}$, $W^s(M \times \mathbb{T})$ decomposes as a direct sum or integral of irreducible, unitary, Sobolev subspaces $H_{m,\mu}^s$, where $H_{m,\mu}^s$ is the restriction of $W^s(M \times \mathbb{T})$ to $H_{m,\mu}$ and has inner product $\langle \cdot, \cdot \rangle_{W^s(M \times \mathbb{T})}$.

For any irreducible component $H_{m,\mu}$, the distributional dual space to $H_{m,\mu}^s$ is denoted $H_{m,\mu}^{-s} := (H_{m,\mu}^s)'$. The subspace of smooth vectors in $H_{m,\mu}$ is denoted $H_{m,\mu}^\infty := \bigcap_{s \geq 0} H_{m,\mu}^s$, and its distributional dual space is denoted

$$H_{m,\mu}^{-\infty} := (H_{m,\mu}^\infty)' = \bigcup_{s \geq 0} H_{m,\mu}^{-s}.$$

In completely analogous fashion, for any $r, s \geq 0$, the foliated Sobolev space $W^{r,s}(M \times \mathbb{T})$ decomposes into a direct sum or integral of irreducible, unitary representation spaces denoted $H_{m,\mu}^{r,s}$. The distributional dual space of $H_{m,\mu}^{r,s}$ is denoted $H_{m,\mu}^{-r,-s}$.

Note that Δ restricts on $L^2(M)$ to the essentially self-adjoint elliptic operator

$$-X^2 - 2(U^2 + V^2).$$

For any $s \in \mathbb{R}$, $W^s(M)$ is defined to be $W^s(M \times \mathbb{T}) \cap L^2(M)$, and it is endowed with an inner product that is obtained from $W^s(M \times \mathbb{T})$. In completely analogous fashion, the Sobolev space $W^s(M)$ decomposes into a direct sum or integral of irreducible, unitary subspaces $H_\mu^s := H_{0,\mu}^s$.

The space of smooth functions on M is defined to be $C^\infty(M) = C^\infty(M \times \mathbb{T}) \cap L^2(M)$, and its distributional dual space is $\mathcal{E}'(M)$. We let $H_\mu^\infty := H_{0,\mu}^\infty$ and $H_\mu^{-\infty} := H_{0,\mu}^{-\infty}$.

Similarly, the foliated space $\widehat{W}^{r,s}(M)$ decomposes into a direct sum or integral of irreducible, unitary subspaces $\widehat{H}_\mu^{r,s} := \widehat{H}_{0,\mu}^{r,s}$.

For any $L > 0$ and $N \in \mathbb{Z}^+$, let

$$\begin{aligned} \mathcal{J}^{s,L}(\Gamma) &:= \{ \mathcal{D} \in W^{-s}(M) : h_L \mathcal{D} = \mathcal{D} \}, \\ \mathcal{J}^L(\Gamma) &:= \{ \mathcal{D} \in \mathcal{E}'(M) : h_L \mathcal{D} = \mathcal{D} \} \end{aligned}$$

be the space of h_L -invariant distributions of order $s \geq 0$ and infinity in $W^{-s}(M)$ and $\mathcal{E}'(M)$, respectively. Let

$$\mathcal{J}^0(\Gamma) := \{ \mathcal{D} \in \mathcal{E}'(M) : U \mathcal{D} = \mathcal{D} \}$$

be the space of invariant distributions for the horocycle flow in $\mathcal{E}'(M)$, and let

$$\begin{aligned} \text{Ann}^{s,L}(\Gamma) &:= \{ f \in W^s(M) : \mathcal{D}(f) = 0 \text{ for all } \mathcal{D} \in \mathcal{J}^{s,L}(\Gamma) \}, \\ \text{Ann}^L(\Gamma) &:= \{ f \in C^\infty(M) : \mathcal{D}(f) = 0 \text{ for all } \mathcal{D} \in \mathcal{J}^L(\Gamma) \}. \end{aligned}$$

We know from Theorem 1.2 of [23] that these are the spaces of coboundaries of Sobolev regularity $s \geq 0$ and ∞ for the horocycle map h_L .

Let

$$\mathcal{J}^0(\Gamma) := \{ \mathcal{D} \in \mathcal{E}'(M) : U \mathcal{D} = \mathcal{D} \}$$

and, by Theorem 1.2 of [6], the space of smooth coboundaries for the horocycle flow is

$$\text{Ann}^0(\Gamma) := \{ f \in C^\infty(M) : \mathcal{D}(f) = 0 \text{ for all } \mathcal{D} \in \mathcal{J}^0(\Gamma) \}.$$

By Theorem 1.1 of [23], Theorem 1.1 of [6] and Theorem 2.1 above, the space $\mathcal{J}^{\infty,L}(M)$ is described as follows.

Theorem 2.6. *Let σ_{pp} be the spectrum of the Laplace-Beltrami operator Δ on $L^2(M)$. Then in any Sobolev structure $W^{-s}(M)$, for $s > 0$, there is a splitting*

$$\mathcal{J}^L(\Gamma) = \mathcal{J}^0(M) \oplus \mathcal{J}^{L,\text{twist}}(\Gamma),$$

where we have $\mathcal{J}^{L,\text{twist}}(\Gamma) \subset W^{-(1/2+)}(M)$ and for each irreducible, unitary space H , the space $\mathcal{J}^{L,\text{twist}}(\Gamma) \cap H^{-(1/2+)}$ has infinite, countable dimension.

The space $\mathcal{J}^0(\Gamma)$ is described in Theorem 1.1 of [6] as follows: It has infinite, countable dimension. It is a direct sum of the trivial representation \mathcal{J}_{vol} and irreducible, unitary representations \mathcal{J}_μ belonging to the principal series, the complementary series, the discrete series and the mock discrete series.

Specifically,

- The space \mathcal{I}_{vol} is spanned by the $\text{SL}(2, \mathbb{R})$ -invariant volume;
- For $0 < \mu < 1$, there is a splitting $\mathcal{I}_{\mu} = \mathcal{I}_{\mu}^{+} \oplus \mathcal{I}_{\mu}^{-}$, where $\mathcal{I}_{\mu}^{\pm} \subset W^{-s}(M)$ if and only if $s > \frac{1 \pm \sqrt{1-\mu}}{2}$, and each subspace has dimension equal to the multiplicity of $\mu \in \text{spec}(\square)$;
- If $\mu \geq 1$, then $\mathcal{I}_{\mu} \subset W^{-s}(M)$ if and only if $s > 1/2$, and it has dimension equal to twice the multiplicity of $\mu \in \text{spec}(\square)$;
- If $\mu = -n^2 + 2n$ for $n \in \mathbb{Z}^{+}$, then $\mathcal{I}_{\mu} \subset W^{-s}(M)$ if and only if $s > n/2$ and it has dimension equal to twice the rank of the space of holomorphic sections of the n_{th} power of the canonical line bundle over M .

The description of $\mathcal{I}^{s,L}(\Gamma)$ is given by $\mathcal{I}^{s,L}(\Gamma) := \mathcal{I}^L(\Gamma) \cap W^{-s}(M)$, where $\mathcal{I}^{s,0}(\Gamma) := \mathcal{I}^0(\Gamma) \cap W^{-s}(M)$ and $\mathcal{I}^{s,L,\text{twist}}(\Gamma) := \mathcal{I}^{L,\text{twist}}(\Gamma) \cap W^{-s}(M)$.

By Theorem 1.4 of [6], the space \mathcal{I}_{μ} has a basis of generalized eigendistributions for the geodesic flow. Consequently, the projection of the ergodic sum for h_L to $\mathcal{I}^{s,0}(M)$ is controlled by estimating the decay of invariant distributions under the action of the geodesic flow, see Section 5 of [6] and Proposition 7.3 of [23].

In contrast, there is no subspace of $\mathcal{I}^{s,L,\text{twist}}(\Gamma)$ that is invariant under the geodesic flow. The projection of the ergodic sum to $\mathcal{I}^{s,L,\text{twist}}(\Gamma)$ will be controlled by the ergodic average of the twisted horocycle flow.

Let $\mathcal{I}_{\mu}^{s,0} := \mathcal{I}_{\mu} \cap W^{-s}(M)$ be the space of horocycle flow-invariant distributions in \mathcal{I}_{μ} of order s . For any $\mu \in \text{spec}(\square)$, set

$$s_{\mu}^{\pm} := \begin{cases} \frac{1 \pm \text{Re} \sqrt{1-\mu}}{2} & \text{if } \mu > 0; \\ n/2 & \text{if } \mu = -n^2 + 2n \text{ and } n \in \mathbb{Z}^{+}. \end{cases}$$

Theorem 2.7 on the rate of equidistribution of horocycle maps is stated in terms of anisotropic Sobolev norms, which we presently describe. The operator $-U^2$ is non-negative and essentially self-adjoint. Then for any $a \geq 0$, $(I - U^2)^{a/2}$ is defined by the spectral theorem. Then for any $a \geq 0$ and for any $r, s \geq 0$, we define $W^{r,s,a}(M)$ to be the Sobolev subspace that is the maximal domain of the operator

$$((I + \square^2)^{r/2} (I + \square^2 + \widehat{\Delta}^2)^{s/2} (I - U^2)^{a/2}$$

on $L^2(M)$, which is endowed with the inner product

$$\langle f, g \rangle_{W^{r,s,a}(M)} := \langle (I - U^2)^{a/4} f, (I - U^2)^{a/4} g \rangle_{r,s}.$$

We denote the corresponding norm by

$$\|f\|_{r,s,a} := \langle f, f \rangle_{W^{r,s,a}(M)}^{1/2}.$$

The dual space of $W^{r,s,a}(M)$ is denoted $W^{-r,-s,-a}(M)$ and has the corresponding dual norm denoted $\|\cdot\|_{-r,-s,-a}$. Observe that for any $r, s \geq 0$ and $a > 0$, we have the continuous embeddings

$$(24) \quad W^{0,r+s+a,r+s+a}(M) \subset W^{r+s+a}(M) \subset W^{r,s,a}(M) \subset \widehat{W}^{r,s}(M).$$

For all $(x, L, N) \in M \times \mathbb{R}^+ \times \mathbb{N}$ we define the constants

$$\begin{aligned} C_\Gamma(x, L, N) &:= C_\Gamma(h_{-L/2}(x), NL) + C_\Gamma(h_{L(N-1/2)}(x), NL); \\ D_\Gamma(x, L, N) &:= e^{d_M(h_{-L/2}(x))} + e^{d_M(h_{L(N-1/2)}(x))}. \end{aligned}$$

Theorem 2.7. *For any $s > 2$, $a > 2$ and, $r \geq 5s - 3$, and for any $\varepsilon > 0$, there are constants $C_{r,s,a,\varepsilon} := C_{r,s,a,\varepsilon}(\Gamma) > 0$ and $C_{r,s,\varepsilon} := C_{r,s,\varepsilon}(\Gamma) > 0$ such that the following holds. For any $(x, L, N) \in M \times \mathbb{R}^+ \times \mathbb{Z}^+$, there is a decomposition of the ergodic sum of the time- L horocycle map h_L as a distribution in $W^{-r,-s,-a}(M)$ as follows. We have*

$$\sum_{n=0}^{N-1} (h_{Ln}(x))^* = \mathcal{D}_{x,N,L,r,s,a}^0 + \mathcal{D}_{x,N,L,r,s,a}^{\text{twist}} + \mathcal{R}_{x,N,L,r,s,a}.$$

The distribution $\mathcal{D}_{x,N,L,r,s,a}^0$ is invariant under the horocycle flow, the distribution $\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}$ is invariant under the time- L horocycle map (but not under the horocycle flow), and the distribution $\mathcal{R}_{x,N,L,r,s,a}$ belongs to the orthogonal subspace $\mathcal{J}^{s,L}(\Gamma)^\perp$ of the space $\mathcal{J}^{s,L}(\Gamma)$ of invariant distributions for the time- L horocycle map. For all $f \in W^{2s+a+1+\varepsilon}(M)$ and for all $(x, L, N) \in M \times \mathbb{R}^+ \times \mathbb{Z}^+$ such that $NL \geq e$ the following estimates hold:

$$\begin{aligned} & \left| \mathcal{D}_{x,N,L,r,s,a}^0(f) - \frac{1}{L} \int_0^{NL} f \circ h_t(x) dt \right| \\ & \leq C_{r,s,a,\varepsilon} D_\Gamma(x, L, N) \frac{1 + L^{2s+2+\varepsilon}}{L} \|f\|_{r,s,s+a}; \\ & |\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}(f)| \leq C_{r,s,\varepsilon} (1 + L^{8s+\varepsilon}) C_\Gamma(x, L, N) (NL)^{\frac{5}{6}} \log^{\frac{1}{2}}(NL) \|f\|_{r,s,1+\varepsilon} \\ & \quad + C_{r,s,a,\varepsilon} D_\Gamma(x, L, N) (1 + L^{2s+2+\varepsilon}) \|f\|_{r,s,s+a+1+\varepsilon}; \\ & |\mathcal{R}_{x,N,L,r,s,a}(f)| \leq C_{r,s,a,\varepsilon} D_\Gamma(x, L, N) \frac{1 + L^{2+\varepsilon}}{L} \|f\|_{r,s,a}. \end{aligned}$$

As in Corollary 2.5, the above theorem can be improved for $L \geq 1$ by a geodesic scaling argument. For all $(x, L, N) \in M \times \mathbb{R}^+ \times \mathbb{N}$ we define the constants

$$\begin{aligned} \tilde{C}_\Gamma(x, L, N) &:= C_\Gamma(a_{\log L}(x), 1, N); \\ \tilde{D}_\Gamma(x, L, N) &:= D_\Gamma(a_{\log L}(x), 1, N). \end{aligned}$$

Corollary 2.8. *For every $s > 14$ and every $\varepsilon > 0$, there is a constant $C'_{s,\varepsilon} := C'_{s,\varepsilon}(\Gamma) > 0$ such that the following holds. For every $L \geq 1$, the following bounds on the ergodic sums for horocycle maps holds: for every $(x, N) \in M \times \mathbb{N}$ and for every function $f \in W^s(M)$, we have,*

$$\begin{aligned} (25) \quad & \left| \sum_{n=0}^{N-1} f \circ h_{Ln}(x) - \frac{1}{L} \int_0^{NL} f \circ h_t(x) dt \right| \leq C'_{s,\varepsilon} \|f\|_s \\ & \times \left(\tilde{C}_\Gamma(x, L, N) L^{1/6+\varepsilon} (NL)^{\frac{5}{6}} \log^{\frac{1}{2}} N + \tilde{D}_\Gamma(x, L, N) (1 + L^{5+\varepsilon}) \right). \end{aligned}$$

Proof. For any $\varepsilon > 0$, let $s, a \in (2, 2 + \varepsilon/4)$, and $r \in (7, 7 + \varepsilon/4)$. Then for all $f \in W^{14+2\varepsilon}(M)$,

$$(26) \quad \|f\|_{r,s,a} \leq \|f\|_{r,s,s+a} \leq \|f\|_{r,s,s+a+1+\varepsilon} \leq \|f\|_{14+2\varepsilon}.$$

For any $t \in \mathbb{R}$, we have $a_{\log L}^{-1} \circ h_t \circ a_{\log L} = h_{Lt}$. Then

$$\begin{aligned} \sum_{n=0}^{N-1} f \circ h_{Ln}(x) &= \sum_{n=0}^{N-1} (f \circ a_{\log L}^{-1}) \circ h_n(a_{\log L}(x)); \\ \frac{1}{L} \int_0^{NL} f \circ h_t(x) dt &= \int_0^N f \circ a_{\log L}^{-1} \circ h_t(a_{\log L}(x)) dt. \end{aligned}$$

Now by Theorem 2.7 in the case $L = 1$ we have a decomposition

$$\begin{aligned} \sum_{n=0}^{N-1} (f \circ a_{\log L}^{-1}) \circ h_n(a_{\log L}(x)) &= \mathcal{D}_{a_{\log L}(x), N, 1, r, s, a}^0 (f \circ a_{\log L}^{-1}) \\ &\quad + \mathcal{D}_{a_{\log L}(x), N, 1, r, s, a}^{\text{twist}} (f \circ a_{\log L}^{-1}) + \mathcal{R}_{a_{\log L}(x), N, 1, r, s, a} (f \circ a_{\log L}^{-1}), \end{aligned}$$

such that the bounds stated in Theorem 2.7 hold: for all $(\tilde{x}, N) \in M \times \mathbb{Z}^+$ and for all $\tilde{f} \in W^{r,s,a}(M)$ we have

$$\begin{aligned} |\mathcal{D}_{\tilde{x}, N, 1, r, s, a}^0(\tilde{f}) - \int_0^N \tilde{f} \circ h_t(x) dt| &\leq C_{r,s,a,\varepsilon} D_\Gamma(\tilde{x}, 1, N) \|\tilde{f}\|_{r,s,s+a}; \\ |\mathcal{D}_{\tilde{x}, N, 1, r, s, a}^{\text{twist}}(\tilde{f})| &\leq C_{r,s,\varepsilon} C_\Gamma(\tilde{x}, 1, N) N^{\frac{5}{6}} \log^{\frac{1}{2}}(e + N) \|\tilde{f}\|_{r,s,1+\varepsilon}; \\ &\quad + C_{r,s,a,\varepsilon} D_\Gamma(\tilde{x}, 1, N) \|\tilde{f}\|_{r,s,s+a+1+\varepsilon}; \\ |\mathcal{R}_{\tilde{x}, N, 1, r, s, a}(\tilde{f})| &\leq C_{r,s,a,\varepsilon} D_\Gamma(\tilde{x}, 1, N) \|\tilde{f}\|_{r,s,a}. \end{aligned}$$

By the above bounds for $\tilde{x} = a_{\log L}(x)$ and $\tilde{f} = f \circ a_{\log L}^{-1}$ we derive the estimate

$$\begin{aligned} & \left| \sum_{n=0}^{N-1} f \circ h_{Ln}(x) - \frac{1}{L} \int_0^{NL} f \circ h_t(x) dt \right| \\ & \leq 3C_{r,s,a,\varepsilon} D_\Gamma(a_{\log L}(x), 1, N) \|f \circ a_{\log L}^{-1}\|_{r,s,s+a+1+\varepsilon} \\ & \quad + C_{r,s,\varepsilon} C_\Gamma(a_{\log L}(x), 1, N) N^{\frac{5}{6}} \log^{\frac{1}{2}}(e + N) \|f \circ a_{\log L}^{-1}\|_{r,s,1+\varepsilon}. \end{aligned}$$

The proof of the main estimate in Theorem 1.2 then follows from straightforward scaling estimates for the Sobolev norms under the action of the geodesic flow: for all (r, s, a) , for all functions $f \in W^{r,s,a}(M)$ and for all $L > 0$, we have

$$\|f \circ a_{\log L}^{-1}\|_{r,s,a} \leq (1 + L^{-s})(1 + L^a) \|f\|_{r,s,a}.$$

Since we have chosen $s, a \in (2, 2 + \varepsilon/4)$ it follows that

$$s + a + 1 + \varepsilon \leq 5 + \varepsilon$$

so that, for all functions $f \in W^{r,s,a}(M)$ and for all $L \geq 1$ we have

$$\|f \circ a_{\log L}^{-1}\|_{r,s,1+\varepsilon} \leq 4L^{1+\varepsilon} \quad \text{and} \quad \|f \circ a_{\log L}^{-1}\|_{r,s,s+a+1+\varepsilon} \leq 4L^{5+\varepsilon}.$$

The argument is therefore completed. \square

Proof of Theorem 1.2. For $NL \leq e$ the statement is immediate since for continuous functions Riemann sums approximate the integral. We can therefore assume that $NL \geq e$

We first prove the bound in formula (6) when $L \leq e$. The triangle inequality gives

$$d_M(h_{-L/2}(x)) - d_M(x) \leq e.$$

Hence, there is a constant $C > 0$ such that

$$(27) \quad D_\Gamma(x, L, N) \leq C(e^{d_M(x)} + e^{d_M(h_{NL}(x))}).$$

If $x, h_{NL}(x) \in M_{A,Q}$, we immediately have that $D_\Gamma(x, L, N) \leq 2Ce^Q$ and, by the definition of $C_\Gamma(x, L, N)$, there is a constant $C'_{\Gamma,A,Q} > 0$ such that

$$C_\Gamma(x, L, N) \leq C'_{\Gamma,A,Q}(NL)^{\frac{2A}{1-A}}.$$

Then (6) for $L \leq e$ follows from Theorem 2.7.

By the logarithm law for geodesics, there is a measurable, finite almost everywhere function $C_\varepsilon : M \rightarrow \mathbb{R}^+$ such that

$$C_\Gamma(x, N, L) \leq [C_\varepsilon(x) + C_\varepsilon(h_{NL}(x))](1 + \log^{1+\varepsilon}(NL)).$$

Now for any $s' > 2, a > 2, r \geq 5s' - 3$ and $\varepsilon' > 0$ such that $r + s' + a + \varepsilon' < s$, there is a constant $C_s := C_s(\Gamma) > 0$ such that for all constants $C_{r,s',a,\varepsilon}(\Gamma) > 0$ given in Theorem 2.7, we have

$$C_{r,s',a,\varepsilon}(\Gamma) \leq C_s.$$

Then there is a measurable, finite almost everywhere function $C'_{s,\varepsilon} : M \rightarrow \mathbb{R}^+$ given by

$$C'_{s,\varepsilon}(x) \geq C_s + C_{\varepsilon'}(x) + e^{d_M(x)}.$$

The statement (7) for $L \leq e$ follows from this.

When $L \geq e$, the statements follow by similar arguments with Corollary 2.8 in place of Theorem 2.7. In fact we have

$$\begin{aligned} \tilde{C}_\Gamma(x, L, N) &= C_\Gamma(a_{\log L}(x), 1, N) \\ &= C_\Gamma(h_{-1/2} \circ a_{\log L}(x), N) + C_\Gamma(h_{N-1/2} \circ a_{\log L}(x), N) \\ \tilde{D}_\Gamma(x, L, N) &= D_\Gamma(a_{\log L}(x), 1, N) \\ &= e^{d_M(h_{-1/2} \circ a_{\log L}(x))} + e^{d_M(h_{N-1/2} \circ a_{\log L}(x))}. \end{aligned}$$

Let us assume that x and $h_{NL}(x) \in M_{A,Q}$ then we have

$$\begin{aligned} d_M(a_t \circ h_{-1/2} \circ a_{\log L}(x)) &= d_M(h_{-e^{-t}/2} \circ a_{t+\log L}(x)) \\ &\leq At + A \log L + Q + 1/2 \\ d_M(a_t \circ h_{N-1/2} \circ a_{\log L}(x)) &= d_M(h_{-e^{-t}/2} \circ a_{t+\log L}(h_{NL}(x))) \\ &\leq At + A \log L + Q + 1/2. \end{aligned}$$

It follows by Lemma 6.5 that there exist constants $C_{\Gamma,Q}, D_Q > 0$ such that

$$\begin{aligned} \tilde{C}_\Gamma(x, L, N) &\leq C_{\Gamma,Q}(NL)^{\frac{2A}{1-A}}; \\ \tilde{D}_\Gamma(x, L, N) &\leq D_Q L^A. \end{aligned}$$

The bound in formula (6) for $L \geq e$ then follows immediately from Corollary 2.8 and from the above inequalities.

Let us then assume that x and $h_{NL}(x) \in \tilde{M}_{A,Q}$, that is, that there exist constants $A > 1/2$ and $Q > 0$ such that $d_M(a_y(x)), d_M(a_y(h_{NL}(x))) \leq A \log y + Q$ for all $y \geq 1$. For all $L \geq e$ and all $t \geq 0$ we have

$$\begin{aligned} d_M(a_t \circ h_{-1/2} \circ a_{\log L}(x)) &= d_M(h_{-e^{-t}/2} \circ a_{t+\log L}(x)) \\ &\leq A \log(t + \log L) + Q + 1/2 \\ d_M(a_t \circ h_{N-1/2} \circ a_{\log L}(x)) &= d_M(h_{-e^{-t}/2} \circ a_{t+\log L}(h_{NL}(x))) \\ &\leq A \log(t + \log L) + Q + 1/2, \end{aligned}$$

thus by Lemma 6.5 it follows that there exists a constant $C_{\Gamma,A} > 0$ such that

$$\begin{aligned} \tilde{C}_{\Gamma}(x, L, N) &\leq C_{\Gamma,A} e^{2Q} (1 + Q + \log(NL))^{2A}; \\ \tilde{D}_{\Gamma}(x, L, N) &\leq 2A \log \log L + 2Q + 1. \end{aligned}$$

The bound in formula (7) for $L \geq e$ then follows immediately from Corollary 2.8 and from the above inequalities.

Next, for the estimate (8), again $s' > 2, a > 2$ and $r \geq 5s' - 3$ be such that $r + s' + a < s$. Let $f \in W^s(M)$ be a coboundary for h_L of zero average. Because $\mathcal{D}_{x,N,L,r,s',a}^0$ and $\mathcal{D}_{x,N,L,r,s',a}^{\text{twist}}$ are invariant under h_L , we have

$$\mathcal{D}_{x,N,L,r,s',a}^0(f) = \mathcal{D}_{x,N,L,r,s',a}^{\text{twist}}(f) = 0$$

Then by the decomposition in Theorem 2.7, it follows that

$$\begin{aligned} \left| \sum_{k=0}^{N-1} f \circ h_{Lk}(x) \right| &= |\mathcal{R}_{x,N,L,r,s',a}(f)| \\ &\leq C_{s,\varepsilon} (e^{d_M(h_{-L/2}(x))} + e^{d_M(h_{L(N-1/2)}(x))}) \frac{1 + L^{2+\varepsilon}}{L} \|f\|_{r,s',a} \\ &\leq C_{s,\varepsilon} (e^{d_M(h_{-L/2}(x))} + e^{d_M(h_{L(N-1/2)}(x))}) \frac{1 + L^{2+\varepsilon}}{L} \|f\|_s. \end{aligned}$$

Finally, the statement for M compact is immediate from Theorem 2.7. \square

3. TWISTED HOROCYCLE FLOWS: COHOMOLOGICAL EQUATIONS

In this section we prove Theorem 2.1 on invariant distributions and Theorem 2.2 on solutions of the cohomological equation for the twisted horocycle flow. The argument is carried out in irreducible unitary representations of $SL(2, \mathbb{R}) \times \mathbb{T}$ of every unitary type. Our bounds are proved with respect to rescaled foliated Sobolev norms introduced to prove the effective equidistribution theorem, Theorem 2.3, with the optimal exponent within reach of our method.

It will often be convenient to use the representation parameter $v = \sqrt{1 - \mu}$ in place of the spectral Casimir parameter $\mu \in \mathbb{R}$.

3.1. Rescaled Sobolev norms. For each $\mathcal{T} \geq 1$, let $X_{\mathcal{T}}$ and $V_{\mathcal{T}}$ be the rescaled vector fields, defined as follows:

$$(28) \quad X_{\mathcal{T}} = \mathcal{T}^{-1/3}X \quad \text{and} \quad V_{\mathcal{T}} = \mathcal{T}^{-2/3}V.$$

Let $(\mathcal{T}, M \times \mathbb{T})$ be the Riemannian manifold $M \times \mathbb{T}$ endowed with the metric that makes the ordered basis of vector fields $(K, \mathcal{T}(U + K), X_{\mathcal{T}}, V_{\mathcal{T}})$ of the Lie algebra $\mathbb{R} \times \mathfrak{sl}(2, \mathbb{R})$ orthonormal. Let $\{\phi_t^{\lambda, \mathcal{T}}\}_{t \in \mathbb{R}}$ on $M \times \mathbb{T}$ be the flow that is generated by $\mathcal{T}(U + \lambda K)$. The rescaled foliated Laplacian is the essentially self-adjoint non-negative operator

$$\widehat{\Delta}_{\mathcal{T}} := -K^2 - X_{\mathcal{T}}^2 - V_{\mathcal{T}}^2.$$

The *rescaled foliated Sobolev spaces* $\widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T})$ are defined as follows. Let $\square_{\mathcal{T}}$ denote the rescaled Casimir operator

$$\square_{\mathcal{T}} := \mathcal{T}^{-2/3}\square.$$

For $r \geq 0, s \geq 0$, let $\widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T})$ be the maximal domain of the operator $(I + \square^2)^{r/2}(I + \square_{\mathcal{T}}^2 + \widehat{\Delta}_{\mathcal{T}}^2)^{s/2}$ on $L^2(M \times \mathbb{T})$ with inner product

$$\langle F, G \rangle_{\widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T})} := \langle (I + \square^2)^{r/2}(I + \square_{\mathcal{T}}^2 + \widehat{\Delta}_{\mathcal{T}}^2)^{s/2}F, G \rangle_{L^2(M \times \mathbb{T})}.$$

The norm on the space $\widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T})$ is defined as

$$\|F\|_{r,s;\mathcal{T}} := \langle F, F \rangle_{\widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T})}$$

Let $\widehat{W}_{\mathcal{T}}^{-r,-s}(M \times \mathbb{T}) = \left(\widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T})\right)'$ be its distributional dual. Notice that by definition, for any $r, s \geq 0$, the Sobolev space we have the continuous embeddings

$$\begin{aligned} W^{r+s}(M \times \mathbb{T}) &\subset \widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T}), \\ \widehat{W}_{\mathcal{T}}^{-r,-s}(M \times \mathbb{T}) &\subset W^{-(r+s)}(M \times \mathbb{T}). \end{aligned}$$

In completely analogous fashion to the decomposition of $W^s(M \times \mathbb{T})$, we have that the space $\widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T})$ decomposes into a direct sum or a direct integral of irreducible, unitary, Sobolev subspaces $\widehat{H}_{m,\mu,\mathcal{T}}^{r,s}$, where $\widehat{H}_{m,\mu,\mathcal{T}}^{r,s}$ is the intersection of $\widehat{W}_{\mathcal{T}}^{r,s}(M \times \mathbb{T})$ to $H_{m,\mu}$ and is endowed with the inner product $\langle \cdot, \cdot \rangle_{r,s;\mathcal{T}}$. By definition, since the Casimir operator acts as a constant multiple of the identity on each irreducible subspace $H_{m,\mu}$, it follows that for all $r, r', s \geq 0$, the spaces $\widehat{H}_{m,\mu,\mathcal{T}}^{r,s}$ and $\widehat{H}_{m,\mu,\mathcal{T}}^{r',s}$ coincide as vector spaces, but are endowed with different inner products and norms.

We simplify our notation. For $H := H_{m,\mu}$ and for $r, s > 0$, the distributional dual space to $\widehat{H}_{m,\mu,\mathcal{T}}^{r,s} := \widehat{H}_{m,\mu,\mathcal{T}}^{r,s}$ is denoted $\widehat{H}^{-r,-s} := \left(\widehat{H}^{r,s}\right)'$. The space of smooth vectors in H in this foliated sense is denoted $\widehat{H}^{\infty} := \bigcap_{s \geq 0} \widehat{H}_{\mathcal{T}}^{0,s}$, and its distributional dual space is denoted $\widehat{H}^{-\infty} := \left(\widehat{H}^{\infty}\right)' = \bigcup_{s \geq 0} \widehat{H}_{\mathcal{T}}^{0,-s}$. Our convention is justified since all the above spaces do not depend, as topological vector spaces, on the scaling parameter $\mathcal{T} > 0$ (but of course the rescaled foliated Sobolev norms do depend on the scaling parameter).

3.2. Principal and complementary series. Let H_μ be the line model of an irreducible unitary representation of the principal or complementary series (see Appendix A). By Plancherel's theorem, a computation left for Appendix B shows

Lemma 3.1. *Then there is a constant $C > 0$ such that, for all $f \in H_\mu$,*

$$\|f\|_0^2 = C \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{-\text{Re } \nu} d\xi.$$

Notice that the Fourier transform of the X and V operators are

$$(29) \quad \hat{X} := (1 - \nu) + 2\xi \frac{\partial}{\partial \xi}, \quad \hat{V} := -i \left((1 - \nu) \frac{\partial}{\partial \xi} + \xi \frac{\partial^2}{\partial \xi^2} \right).$$

In fact, the above formulas hold by the standard property of the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R})$ of rapidly decaying smooth functions, hence they hold by duality on the space $\mathcal{S}'(\mathbb{R})$ of tempered distributions.

By Lemma 3.1, it follows that there is also a formula for the rescaled foliated Sobolev norms of functions in Fourier transform.

Let us adopt the following notation. For $\mu \in \text{spec}(\square)$, define $\mu_{\mathcal{T}} := \mathcal{T}^{-2/3} \mu$ so that

$$\text{spec}(\square_{\mathcal{T}}) = \{\mu_{\mathcal{T}} | \mu \in \text{spec}(\square)\}.$$

Lemma 3.2. *Let $H := H_{m,\mu}$ and let $r, s \geq 0$. For all $\mathcal{T} \geq 1$ and for all $f \in \widehat{H}^\infty$, we have*

$$|f|_{r,s;\mathcal{T}}^2 = C(1 + \mu^2)^{\frac{s}{2}} \int_{\mathbb{R}} |[I + \mu_{\mathcal{T}}^2 + (m^2 I - \hat{X}_{\mathcal{T}}^2 - \hat{V}_{\mathcal{T}}^2)^2]^{\frac{s}{4}} \hat{f}(\xi)|^2 \frac{d\xi}{|\xi|^{\text{Re } \nu}}.$$

3.2.1. Invariant distributions. Let $m \in \mathbb{Z}$ and $\mu > 0$. Let H_μ be the line model of an irreducible, unitary representation in the principal or complementary series of $\text{SL}(2, \mathbb{R})$, and let $H_{m,\mu}$ the corresponding model for an irreducible unitary representation of $\text{SL}(2, \mathbb{R}) \times \mathbb{T}$ of parameters (m, μ) . For any function $f_m \in H_{m,\mu}$ there exists a function $f \in H_\mu$ such that

$$f_m = f \otimes e_m.$$

We formally define the functional $D_{m,\mu}^\lambda$ on $H_{m,\mu}$ by the formula

$$(30) \quad D_{m,\mu}^\lambda(f_m) := \int_{\mathbb{R}} f(t) e^{-it\lambda m} dt.$$

Note that the functional $D_{m,\mu}^\lambda$ on $H_{m,\mu}$ induces a functional $\bar{D}_{m,\mu}^\lambda$ on H_μ so that by definition we formally have that

$$\bar{D}_{m,\mu}^\lambda = \bar{D}_{1,\mu}^{\lambda m}.$$

By the above identity we can reduce all statements about functionals $D_{m,\mu}^\lambda$ on $H_{m,\mu}$ to statements about functionals $\bar{D}_{1,\mu}^{\lambda m}$ on H_μ .

We now show that $D_{m,\mu}^\lambda$ is densely defined on $H_{m,\mu}$. We will use the horocycle flow invariant distribution D_μ^+ that is is (sharply) defined on $H_\mu^{(1+\text{Re } \nu)/2+}$ by

$$D_\mu^+(f) := \lim_{x \rightarrow \infty} \frac{f(x)}{(1 + x^2)^{(1+\nu)/2}}$$

(see Section 3.2 of [6]). It is immediate that any $f \in \text{Ann}(D_\mu^+)$ is also in $L^1(\mathbb{R})$, and hence, $D_{m,\mu}^\lambda(f \otimes e_m)$ is defined by the above formula.

By a standard construction of an orthogonal basis in H_μ , the element $u_0 \in \text{Ann}(U - V)$ given by

$$u_0(x) := (1 + x^2)^{-(1+\nu)/2}$$

is in H_μ^∞ . Integration by parts shows that $D_{m,\mu}^\lambda(u_0 \otimes e_m) \in \mathbb{C}$. Then for any $f_m = f \otimes e_m \in H_{m,\mu}^\infty$, the distribution $D_{m,\mu}^\lambda$ is defined by

$$D_{m,\mu}^\lambda(f_m) := D_\lambda(f_m - D_\mu^+(f)(u_0 \otimes e_m)) + D_\mu^+(f)D_{m,\mu}^\lambda(u_0 \otimes e_m).$$

It follows from Lemma 3.2 of [23] that $D_{m,\mu}^\lambda \in H_{m,\mu}^{-((1+\text{Re } \nu)/2+)}.$

In fact, we have the following stronger result.

Lemma 3.3. *Let $H := H_{m,\mu}$, where $\mu > 0$ and $\lambda m \neq 0$. Then*

$$D_{m,\mu}^\lambda \in \widehat{H}^{0, -(1/2+)}.$$

Proof. The Fourier transform $\widehat{D}_{m,\mu}^\lambda$ of the distribution $D_{m,\mu}^\lambda$ on $H_{m,\mu}^\infty$ is the Dirac mass at $-\lambda m$, that is,

$$\widehat{D}_{m,\mu}^\lambda(\widehat{f} \otimes e_m) = \widehat{f}(-\lambda m), \quad \text{for all } f \in \mathcal{S}(R).$$

The general case can be reduced to the case when $m = 1$, therefore we prove the result in that case. For simplicity of notation, let $D^\lambda := D_{1,\mu}^\lambda$. Let I_λ be any open interval such that $-\lambda \in I_\lambda$ and $0 \notin I_\lambda$. By the Sobolev embedding theorem, it follows that the distribution $\widehat{D}^\lambda \in W^{-s}(I_\lambda)$ for all $s > 1/2$. In fact, the distribution \widehat{D}^λ is a probability measure at $-\lambda$ and by Sobolev embedding theorem $W^s(I_\lambda) \subset C^0(I_\lambda)$ for all $s > 1/2$. By a direct calculation we can prove that for every $k \in \mathbb{N}$, there exist constants $C_{k,v,\lambda}, C'_{k,v,\lambda} > 0$ such that, for any function $\widehat{f} \in C^\infty(I_\lambda)$, we have

$$\left\| \frac{d^k \widehat{f}}{d\xi^k} \right\|_{L^2(I_\lambda)} \leq C_{k,v,\lambda} \sum_{i=0}^k \|\widehat{X}^i \widehat{f}\|_{L^2(I_\lambda)} \leq C'_{k,v,\lambda} |f|_{0,k}.$$

By interpolation it follows that for every $s \geq 0$, there exists a constant $C_s > 0$ such that for all $f \in \widehat{H}^{0,s}$, we have

$$\|\widehat{f}\|_{W^s(I_\lambda)} \leq C_s |f|_{0,s},$$

hence $\widehat{f} \in W^s(I_\lambda)$ whenever $\widehat{f} \in \widehat{H}^s$ for any $s > 1/2$. The statement then follows from the Sobolev embedding theorem, as explained above. \square

3.2.2. Twisted cohomological equations. Let $\mathcal{T} \geq 1$ and $\lambda \in \mathbb{R}^*$. For $s \geq 0$, we study the twisted cohomological equation

$$(31) \quad \mathcal{T}(U + \lambda K)g = f$$

in every irreducible subspace of the Sobolev space $\widehat{W}_{\mathcal{T}}^s(M \times \mathbb{T})$ of the principal and complementary series.

The following a priori bounds for solutions of the cohomological equation hold.

Theorem 3.4. *Let $r, s \geq 0$. There is a constant $C_{r,s} > 0$ such that for any $H = H_{m,\mu}$ with $\mu > 0$ and $m \in \mathbb{Z}/\{0\}$, and for any function $f_m \in \hat{H}^\infty \cap \text{Ann}(D_{m,\mu}^\lambda)$, there is a unique solution $g_m \in H$ satisfying (31), and moreover, for all $\mathcal{T} \geq 1$,*

$$|g_m|_{r,s;\mathcal{T}} \leq \frac{C_{r,s}}{\mathcal{T}^{1/3}} \frac{1 + |\lambda m|^{-s}}{|\lambda m|} |f_m|_{r+s,s+1;\mathcal{T}}.$$

By proceeding formally, we note that f_m and g_m are simple tensors, so we write $f_m = f \otimes e_m$ and we consider a solution $g \otimes e_m \in \hat{H}^\infty$ of the cohomological equation

$$(32) \quad \mathcal{T}(U + \lambda K)g \otimes e_m = f \otimes e_m.$$

In a line model $H_{m,\mu}$ of an irreducible unitary representation of the principal or complementary series, the cohomological equation (32) takes the form

$$\mathcal{T}\left(\frac{d}{dx} + i(\lambda m)\right)g(x) \otimes e_m(t) = f(x) \otimes e_m(t).$$

Then it is enough to prove Sobolev a priori estimates for the solution to the equation

$$(33) \quad \mathcal{T}\left(\frac{d}{dx} + i(\lambda m)\right)g = f.$$

By taking the Fourier transform of both sides of (33), we get that for all $\xi \in \mathbb{R}$,

$$(34) \quad \hat{g}(\xi) = -i \frac{\hat{f}(\xi)}{\mathcal{T}(\xi + \lambda m)}.$$

We observe that $D_{m,\mu}^\lambda(e_m \otimes f) = 0$ if and only if $\hat{f}(-\lambda m) = 0$. In what follows, we will simplify notation, and let

$$m := 1.$$

The estimate for $m \neq 1$ will be derived from this case for the parameter equal to λm . In addition, the general estimate for rescaled equation with respect to the rescaled Sobolev norms will be derived from the non-rescaled case. We will therefore let

$$\mathcal{T} := 1,$$

and we are left to consider the formula for the solution:

$$(35) \quad \hat{g}(\xi) = -i \frac{\hat{f}(\xi)}{(\xi + \lambda)}.$$

Now, for any $\hat{f} \in H_\mu$ and any $\xi \in \mathbb{R}$, let $\hat{f}_\lambda(\xi) = \hat{f}(\lambda \xi)$. Sobolev estimates for the solution to (35) will be obtained in Lemma 3.11 from such estimates for the function \hat{g}_λ given by the equivalent equation

$$(36) \quad \hat{g}_\lambda(\xi) = -i \frac{\hat{f}_\lambda(\xi)}{\lambda(\xi + 1)}.$$

We will now prove estimates for the above equation, and to simplify notation, we drop the subscript λ from \hat{f}_λ and \hat{g}_λ .

To further simplify notation, set $D := D_{1,\mu}^1$. As a first step, we have the following identity

Lemma 3.5. *Under the condition that $D(f) = 0$, that is, $\hat{f}(-1) = 0$, we have*

$$\hat{g}(\xi) = -\frac{i}{\lambda} \int_0^1 \hat{f}'(-1+t(\xi+1))dt.$$

Proof. For $t \in \mathbb{R}$, let $F(t) = \hat{f}(-1+t(\xi+1))$. By the fundamental theorem of calculus

$$\hat{f}(\xi) = F(1) = F(0) + \int_0^1 \frac{dF}{dt}(t)dt = (\xi+1) \int_0^1 \hat{f}'(-1+t(\xi+1))dt.$$

The formula for the solution then follows immediately. \square

We will split our estimates into different regions. Let

$$(37) \quad I = \left[-\frac{3}{2}, -\frac{1}{2}\right].$$

For every $v \in (0, 1) \cup i\mathbb{R}$ and for every subinterval $J \subset \mathbb{R}$ we will adopt the following notation

$$(38) \quad L_v^2(J) = L^2\left(J, \frac{d\xi}{|\xi|^{\text{Rev}}}\right).$$

Then the following holds.

Lemma 3.6. *For every $\alpha \in \mathbb{N}$, there exists a constant $C'_\alpha > 0$ such that*

$$\|\hat{X}^\alpha \hat{g}\|_{L_v^2(\mathbb{R} \setminus I)} \leq C'_\alpha |\lambda|^{-1} \sum_{k=0}^{\alpha} \|\hat{X}^k \hat{f}\|_0.$$

Proof. It is clear that $1/(\xi+1) \in L^\infty(\mathbb{R} \setminus I)$ and there exists a constant $C_1 > 0$ such that

$$\left\| \frac{1}{\xi+1} \right\|_{L^\infty(\mathbb{R} \setminus I)} \leq C_1.$$

It follows immediately from the formula for the solution (36) that

$$\|\hat{g}\|_{L_v^2(\mathbb{R} \setminus I)} \leq C_1 |\lambda|^{-1} \|\hat{f}\|_{L_v^2(\mathbb{R} \setminus I)}.$$

Let us now consider derivatives. We have

$$\hat{X}\hat{g}(\xi) = -i \frac{\hat{X}\hat{f}(\xi)}{\lambda(\xi+1)} + i \frac{2\xi\hat{f}(\xi)}{\lambda(\xi+1)^2}.$$

Since the function $\xi/(\xi+1)^2 \in L^\infty(\mathbb{R} \setminus I)$, there exists a constant $C_2 > 0$ such that

$$\left\| \frac{\xi}{(\xi+1)^2} \right\|_{L^\infty(\mathbb{R} \setminus I)} \leq C_2.$$

It follows that

$$\|\hat{X}\hat{g}\|_{L_v^2(\mathbb{R} \setminus I)} \leq (C_1 + 2C_2) |\lambda|^{-1} (\|\hat{X}\hat{f}\|_0 + \|\hat{f}\|_0).$$

For higher order derivatives, by induction we prove the a Leibniz-type formula. There exists universal constants $(a_\ell^{(\alpha)})$ such that for all functions $f_1, f_2 \in C^\infty(\mathbb{R})$, the following identity holds on \mathbb{R} :

$$(39) \quad \hat{X}^\alpha(f_1 f_2)(\xi) = \sum_{\ell=0}^{\alpha} a_\ell^{(\alpha)} \hat{X}^\ell f_1(\xi) (\hat{X} - (1-v))^{\alpha-\ell} f_2(\xi).$$

In particular, for $f_1(\xi) = \hat{f}(\xi)$ and $f_2(\xi) = 1/(\xi + 1)$ on I , we have

$$\hat{X}^\alpha \hat{g}(\xi) = -\frac{i}{\lambda} \sum_{\ell=0}^{\alpha} a_\ell^{(\alpha)} \hat{X}^\ell \hat{f}(\xi) (2\xi \frac{d}{d\xi})^{\alpha-\ell} (\frac{1}{\xi+1}).$$

Another induction argument leads to bounds of the following form. There exists a constant $C_{\alpha,\ell} > 0$ such that

$$\|(2\xi \frac{d}{d\xi})^{\alpha-\ell} (\frac{1}{\xi+1})\|_{L^\infty(\mathbb{R} \setminus I)} \leq C_{\alpha,\ell}.$$

The stated bound therefore follows. \square

For higher order derivatives of the form $X^\alpha V^\beta$, or equivalently $V^\beta X^\alpha$, on the set $\mathbb{R} \setminus I$ we begin by computing the following Leibniz-type formula.

Lemma 3.7. *For any $\beta \in \mathbb{N}$ there exist universal coefficients $(b_{ijk}^{(\beta)})$ such that for any pair of functions f_1, f_2 we have the formula*

$$(40) \quad \hat{V}^\beta(f_1 f_2) = \sum_{\substack{i+j+m \leq \beta \\ k \leq m}} b_{ijk}^{(\beta)} \left[\left(\frac{d}{d\xi} \right)^m \hat{V}^i f_1 \right] [(\hat{X} - (1-\nu))^k \hat{V}^j f_2].$$

Proof. The proof is by induction. For $\beta = 1$ we have by a direct computation

$$\hat{V}(f_1 f_2) = \hat{V}(f_1) f_2 + f_1 \hat{V}(f_2) - i \left[\frac{d}{d\xi} f_1 \right] [(\hat{X} - (1-\nu)) f_2].$$

The statement is therefore verified in this case. The proof on the induction step is based on the above formula and on the following formulas for commutators:

$$[\hat{V}, \frac{d}{d\xi}] = i \frac{d^2}{d\xi^2} \quad \text{and} \quad [\hat{V}, (\hat{X} - (1-\nu))] = [\hat{V}, \hat{X}] = 2\hat{V}.$$

By the induction hypothesis and by formula (40), it follows that in the formula for $\hat{V}^{\beta+1}(f_1 f_2)$ we have terms of the following three types

$$(41) \quad \begin{aligned} & [\hat{V} \left(\frac{d}{d\xi} \right)^m \hat{V}^i f_1] [(\hat{X} - (1-\nu))^k \hat{V}^j f_2], \\ & \left[\left(\frac{d}{d\xi} \right)^m \hat{V}^i f_1 \right] [\hat{V} (\hat{X} - (1-\nu))^k \hat{V}^j f_2], \\ & \left[\left(\frac{d}{d\xi} \right)^{m+1} \hat{V}^i f_1 \right] [(\hat{X} - (1-\nu))^{k+1} \hat{V}^j f_2]. \end{aligned}$$

In fact, by the first of the above commutation relation, by an induction argument for every $k \in \mathbb{N} \setminus \{0\}$ we have

$$(42) \quad [\hat{V}, \left(\frac{d}{d\xi} \right)^k] = -k \left(\frac{d}{d\xi} \right)^{k+1},$$

hence the first term in the above formula (41) is of the required form.

By the second of the above commutation relation, we derive by induction that for every $k \in \mathbb{N} \setminus \{0\}$ and every $i \in \{0, \dots, k-1\}$ there exists universal constants $C_{k,i} > 0$ such that

$$(43) \quad [\hat{V}, (\hat{X} - (1-\nu))^k] = \sum_{i=0}^{k-1} C_{k,i} (\hat{X} - (1-\nu))^i \hat{V},$$

hence the second term in formula (41) is of the required form.

Finally the third term in formula (41) is already in the required form. Thus the induction step is proved and the argument is complete. \square

Lemma 3.8. *For every $\alpha, \beta \in \mathbb{N}$, there exists a constant $C'_{\alpha,\beta} > 0$ such that*

$$\|\hat{X}^\alpha \hat{V}^\beta \hat{g}\|_{L^2_\nu(\mathbb{R} \setminus I)} \leq \frac{C'_{\alpha,\beta}}{|\lambda|} \sum_{i+j+k \leq \alpha+\beta} |1-\nu|^i \|\hat{X}^j \hat{V}^k \hat{f}\|_0.$$

Proof. For $0 \leq \ell \leq \alpha$, let

$$\phi_{i,m}^{(\ell)}(\xi) := \left(\frac{d}{d\xi}\right)^m \hat{V}^i \left(2\xi \frac{d}{d\xi}\right)^{\alpha-\ell} \left(\frac{1}{\xi+1}\right).$$

By Lemma 3.7 and by formulas (36) and (39), we derive the following

$$(44) \quad (\hat{V}^\beta \hat{X}^\alpha \hat{g})(\xi) = \frac{-i}{\lambda} \sum_{\ell \leq \alpha} \sum_{\substack{i+j+m \leq \beta \\ k \leq m}} a_\ell^{(\alpha)} b_{ijkm}^{(\beta)} \phi_{i,m}^{(\ell)}(\xi) \times [(\hat{X} - (1-\nu))^k \hat{V}^j \hat{X}^\ell \hat{f}](\xi).$$

By an induction argument we can prove that for all $\alpha, \beta \in \mathbb{N}$ there exists a constant $K_{\alpha,\beta} > 0$ such that, for all $0 \leq \ell \leq \alpha$, all $i+m \leq \beta$, we have

$$\|\phi_{i,m}^{(\ell)}\|_{L^\infty(\mathbb{R} \setminus I)} \leq K_{\alpha,\beta} (1 + |1-\nu|^i).$$

By taking into account the commutation relation $[\hat{X}, \hat{V}] = -2\hat{V}$, it follows that for all $\alpha, \beta \in \mathbb{N}$ there exists a constant $K'_{\alpha,\beta} > 0$ such that

$$\|\hat{V}^\beta \hat{X}^\alpha \hat{g}\|_{L^2_\nu(\mathbb{R} \setminus I)} \leq \frac{K'_{\alpha,\beta}}{|\lambda|} \sum_{i+j+k \leq \alpha+\beta} |1-\nu|^i \|\hat{X}^j \hat{V}^k \hat{f}\|_0.$$

The statement then follows, again by the above commutation relation. \square

We then prove bounds on the interval I . The estimates will be based on the integral formula for the solution.

Lemma 3.9. *For every $\alpha \in \mathbb{N}$, there exists a constant $C''_\alpha > 0$ such that*

$$\|\hat{X}^\alpha \hat{g}\|_{L^2_\nu(I)} \leq \frac{C''_\alpha}{|\lambda|} \sum_{k=0}^{\alpha+1} |1-\nu|^{\alpha-k} \|\hat{X}^k \hat{f}\|_0.$$

Proof. By Lemma 3.5 and the Minkowski integral inequality

$$\|\hat{g}\|_{L_v^2(I)} \leq \frac{1}{|\lambda|} \int_0^1 \|\hat{f}'(-1+t(\cdot+1))\|_{L_v^2(I)} dt.$$

For all $t \in [0, 1]$, let $I_t \subset \mathbb{R}$ denote the interval

$$I_t = [-1-t/2, -1+t/2].$$

Since for all $t \in [0, 1]$ and all $\xi \in I_t$ we have

$$(45) \quad 1/2 \leq |\xi| \leq 2,$$

by change of variable we have

$$(46) \quad \|\hat{f}'(-1+t(\cdot+1))\|_{L_v^2(I)} \leq 2^{\text{Rev}} t^{-1/2} \left(\int_{I_t} |\hat{f}'(\xi)|^2 \frac{d\xi}{|\xi|^{\text{Rev}}} \right)^{1/2}.$$

We recall that for the principal series $v \in i\mathbb{R}$, and for the complementary series $v \in (0, 1)$. It follows that there exists a constant $C_3 > 0$ such that

$$\begin{aligned} \|\hat{f}'(-1+t(\cdot+1))\|_{L_v^2(I)} &\leq C_3 t^{-1/2} \|\xi \hat{f}'(\xi)\|_0 \\ &\leq C_3 t^{-1/2} (\|\hat{X}\hat{f} - (1-v)\hat{f}\|_0). \end{aligned}$$

Hence, we get by integration over $t \in [0, 1]$ that

$$\|\hat{g}\|_{L_v^2(I)} \leq \frac{2C_3}{3} |\lambda|^{-1} \|\hat{X}\hat{f} - (1-v)\hat{f}\|_0.$$

For higher order derivatives we compute as follows:

$$\begin{aligned} \hat{X}^\alpha \hat{g}(\xi) &= -\frac{i}{\lambda} \int_0^1 (2\xi \frac{d}{d\xi} + (1-v))^\alpha [\hat{f}'(-1+t(\xi+1))] dt \\ (47) \quad &= -\frac{i}{\lambda} \int_0^1 [(\hat{X} + 2(1-t) \frac{d}{d\xi})^\alpha \hat{f}'](-1+t(\xi+1)) dt \end{aligned}$$

By applying as above the Minkowski integral inequality followed by a change of coordinates and (46), we get

$$\begin{aligned} &\| \int_0^1 [(\hat{X} + 2(1-t) \frac{d}{d\xi})^\alpha \hat{f}'](-1+t(\xi+1)) dt \|_{L_v^2(I)} \\ (48) \quad &\leq \int_0^1 \| [(\hat{X} + 2(1-t) \frac{d}{d\xi})^\alpha \hat{f}'](-1+t(\cdot+1)) \|_{L_v^2(I)} dt \\ &\leq 2^{\text{Rev}} \int_0^1 t^{-1/2} \| (\hat{X} + 2(1-t) \frac{d}{d\xi})^\alpha \hat{f}' \|_{L_v^2(I_t)} dt. \end{aligned}$$

We then observe that for all $\alpha \in \mathbb{N}$ and for $\xi \neq 0$ the following identity holds:

$$(49) \quad (\hat{X} + 2(1-t) \frac{d}{d\xi})^\alpha \frac{d}{d\xi} = [\hat{X} + (1-t) \frac{1}{\xi} (\hat{X} - (1-v))]^\alpha \frac{1}{2\xi} (\hat{X} - (1-v)).$$

By induction for all $k \in \mathbb{N}$ and $j \in \mathbb{Z}^+$, we have

$$(50) \quad \hat{X}^k \left(\frac{1}{\xi^j} \right) = (1-2j-v)^k \left(\frac{1}{\xi^j} \right) \quad \text{and} \quad [\hat{X} - (1-v)]^k \left(\frac{1}{\xi} \right) = 2^k \left(\frac{1}{\xi} \right),$$

hence by the upper bound in formula (45) it follows immediately that

$$\|(\hat{X} - (1 - \nu))^k(1/\xi)\|_{L^\infty(I_t)} = 2^k \|1/\xi\|_{L^\infty(I_t)} \leq 2^{k+1}.$$

Thus by the identity in formula (49) and by the Leibniz-type formula (39) it follows that there exists a constant $C_4(\alpha) > 0$ such that

$$\|(\hat{X} + 2(1 - t)\frac{d}{d\xi})^\alpha \hat{f}'\|_{L_v^2(I_t)} \leq C_4(\alpha) \sum_{i+j \leq \alpha} \|(\hat{X} - (1 - \nu))^{i+1} \hat{X}^j \hat{f}\|_0.$$

The statement then follows from the integral bound in formula (48). \square

For higher order derivatives of the form $X^\alpha V^\beta$, or equivalently $V^\beta X^\alpha$, on the interval I we proceed as above.

Lemma 3.10. *For every $\alpha, \beta \in \mathbb{N}$, there exists a constant $C''_{\alpha,\beta} > 0$ such that*

$$\|\hat{V}^\beta \hat{X}^\alpha g\|_{L_v^2(I)} \leq \frac{C''_{\alpha,\beta}}{|\lambda|} (1 + |\nu|)^\beta \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ i \leq \beta}} \|\hat{V}^i \hat{X}^j (\hat{X} - (1 - \nu))^k \hat{f}\|_0.$$

Proof. By formula (47) we have

$$\hat{X}^\alpha \hat{g}(\xi) = -\frac{i}{\lambda} \int_0^1 [(\hat{X} + 2(1 - t)\frac{d}{d\xi})^\alpha \hat{f}'](-1 + t(\xi + 1)) dt.$$

It follows by a short calculation that, for all $\alpha, \beta \in \mathbb{N}$, the derivatives $\hat{V}^\beta \hat{X}^\alpha \hat{g}$ of the solution \hat{g} of the twisted cohomological equation are given by the formula

$$(51) \quad \frac{(-i)^{\beta+1}}{\lambda} \int_0^1 t^\beta [\hat{V} + (1 - t)\frac{d^2}{d\xi^2}]^\beta [(\hat{X} + 2(1 - t)\frac{d}{d\xi})^\alpha \hat{f}'](-1 + t(\xi + 1)) dt.$$

By the above formula, by Minkowski integral inequality and by change of variables, the norm $\|\hat{V}^\beta \hat{X}^\alpha \hat{g}\|_{L_v^2(I_\lambda)}$ is bounded by the expression

$$(52) \quad \frac{2^{\text{Rev}}}{|\lambda|} \int_0^1 t^{\beta-1/2} \|(\hat{V} + (1 - t)\frac{d^2}{d\xi^2})^\beta (\hat{X} + 2(1 - t)\frac{d}{d\xi})^\alpha \hat{f}'\|_{L_v^2(I_t)} dt$$

We observe that for all $\alpha \in \mathbb{N}$ and for $\xi \neq 0$ the following identity holds:

$$\begin{aligned} & [\hat{V} + (1 - t)\frac{d^2}{d\xi^2}]^\beta [\hat{X} + 2(1 - t)\frac{d}{d\xi}]^\alpha \frac{d}{d\xi} \\ &= \{\hat{V} + (1 - t)\frac{1}{\xi}[i\hat{V} - (1 - \nu)\frac{d}{d\xi}]\}^\beta \\ (53) \quad & \times \{\hat{X} + (1 - t)\frac{1}{\xi}[\hat{X} - (1 - \nu)]\}^\alpha \frac{1}{2\xi}[\hat{X} - (1 - \nu)] \\ &= \{\hat{V} + (1 - t)\frac{1}{\xi}[i\hat{V} - \frac{(1 - \nu)}{2\xi}(\hat{X} - (1 - \nu))]\}^\beta \\ & \times \{\hat{X} + (1 - t)\frac{1}{\xi}(\hat{X} - (1 - \nu))\}^\alpha \frac{1}{2\xi}[\hat{X} - (1 - \nu)] \end{aligned}$$

By induction the following identity holds for all $k \in \mathbb{N}$:

$$(54) \quad \hat{V}^k\left(\frac{1}{\xi}\right) = (-i)^k k! \left(\prod_{j=1}^k (j + \nu) \right) \frac{1}{\xi^{k+1}}.$$

By the Leibniz-type formula (39) and by that of Lemma 3.7, from the identity (53), from formulas (50), (54), by the upper bound in formula (45), it follows that there exists a constant $K_{\alpha,\beta} > 0$ such that for all $t \in [0, 1]$ we have

$$(55) \quad \begin{aligned} & \|(\hat{V} + (1-t)\frac{d^2}{d\xi^2})^\beta (\hat{X} + 2(1-t)\frac{d}{d\xi})^\alpha \hat{f}^\nu\|_{L_\nu^2(I_t)} \\ & \leq K_{\alpha,\beta} (1+|\nu|)^\beta \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ i \leq \beta}} \|\hat{V}^i \hat{X}^j (\hat{X} - (1-\nu))^k \hat{f}\|_{L_\nu^2(I)} \\ & \leq K_{\alpha,\beta} (1+|\nu|)^\beta \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ i \leq \beta}} \|\hat{V}^i \hat{X}^j (\hat{X} - (1-\nu))^k \hat{f}\|_0. \end{aligned}$$

The statement follows from the bound given in formula (52). \square

From the above lemmas we derive the following result.

Lemma 3.11. *For every $\alpha, \beta \geq 0$ there exists a constant $C_{\alpha,\beta}^{(3)} > 0$ such that for all $\lambda \neq 0$, the unique solution $g \in L_\nu^2(\mathbb{R})$ of the equation $(U + i\lambda)g = f \in L_\nu^2(\mathbb{R})$ satisfies the estimate*

$$\begin{aligned} \|V^\beta X^\alpha g\|_0 & \leq \frac{C_{\alpha,\beta}^{(3)}}{|\lambda|} (1+|\lambda|^{-\beta})(1+|\nu|)^\beta \\ & \times \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ j \leq \beta}} |1-\nu|^i \|V^j X^k f\|_0. \end{aligned}$$

Proof. For any $\hat{f} \in L_\nu^2(\mathbb{R})$, define $\hat{f}_\lambda(\xi) := \hat{f}(\lambda\xi)$. Notice that for any $\alpha, \beta \in \mathbb{N}$ and for any $\lambda \in \mathbb{R}^*$, we have the following identity

$$(56) \quad \hat{V}^\beta \hat{X}^\alpha \hat{f}_\lambda = \lambda^\beta (\hat{V}^\beta \hat{X}^\alpha \hat{f})_\lambda,$$

whenever $\hat{V}^\beta \hat{X}^\alpha f$ is defined.

By formula (36) the solution to the cohomological equation $(U + i\lambda)\hat{g} = \hat{f}$ can be rewritten as

$$\hat{g}_\lambda(\xi) = -i \frac{\hat{f}_\lambda(\xi)}{\lambda(\xi + 1)}.$$

Since by definition for any $\hat{f} \in L_\nu^2(\mathbb{R})$ we have $\hat{f} = (\hat{f}_\lambda)_{1/\lambda}$, from (56) and from Lemmas 3.8 and 3.10 it follows that

$$\begin{aligned}
(57) \quad \|\hat{V}^\beta \hat{X}^\alpha \hat{g}\|_0 &= \|\hat{V}^\beta \hat{X}^\alpha (\hat{g}_\lambda)_{1/\lambda}\|_0 \leq |\lambda|^{-\beta} \|(\hat{V}^\beta \hat{X}^\alpha \hat{g}_\lambda)_{1/\lambda}\|_0 \\
&\leq \frac{C_{\alpha,\beta}^{(3)}}{|\lambda|} |\lambda|^{-\beta} (1+|\nu|)^\beta \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ j \leq \beta}} |1-\nu|^i \|(\hat{V}^j \hat{X}^k \hat{f}_\lambda)_{1/\lambda}\|_0 \\
&\leq \frac{C_{\alpha,\beta}^{(3)}}{|\lambda|} |\lambda|^{-\beta} (1+|\nu|)^\beta \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ j \leq \beta}} |\lambda|^j |1-\nu|^i \|\hat{V}^j \hat{X}^k \hat{f}\|_0.
\end{aligned}$$

□

For rescaled Sobolev norms we have a similar statement which can be immediately derived from Lemma 3.11.

Lemma 3.12. *For every $\alpha, \beta \geq 0$ there exists a constant $C_{\alpha,\beta}^{(4)} > 0$ such that for all $\lambda \neq 0$ and for all $\mathcal{T} \geq 1$, the solution $g \in L_v^2(\mathbb{R})$ of the rescaled equation $\mathcal{T}(U + i\lambda)g = f \in L_v^2(\mathbb{R})$ satisfies the following estimate with respect to the rescaled Sobolev norms:*

$$\begin{aligned}
\|V_{\mathcal{T}}^\beta X_{\mathcal{T}}^\alpha g\|_0 &\leq \frac{C_{\alpha,\beta}^{(4)}}{|\lambda| \mathcal{T}^{1/3}} (1+|\nu|)^\beta (1+|\lambda|^{-\beta}) \\
&\quad \times \sum_{i+j+k \leq \alpha+\beta+1} [\mathcal{T}^{-1/3} |1-\nu|]^i \|V_{\mathcal{T}}^j X_{\mathcal{T}}^k f\|_0.
\end{aligned}$$

Proof of Theorem 3.4. For $r, s \in \mathbb{N}$ even integers, since the Casimir operator \square takes the value $\mu = 1 - \nu^2$ on any irreducible, unitary representation $H_{m,\mu}$, by expanding the operator $(I + \square^2)^{r/2} (I + \square_{\mathcal{T}}^2 + \hat{\Delta}_{\mathcal{T}}^2)^{s/2}$ into a polynomial expression in $X_{\mathcal{T}}$ and $V_{\mathcal{T}}$, and by the commutation relations we derive that there exists a constant $C_{r,s}^{(0)} > 0$ such that

$$(58) \quad |g|_{r,s;\mathcal{T}} \leq C_{r,s}^{(0)} (1+\mu^2)^{r/4} \sum_{k+m+\alpha+\beta \leq s} [\mathcal{T}^{-1/3} |1-\nu|]^k \|K^m V_{\mathcal{T}}^\beta X_{\mathcal{T}}^\alpha g\|_0.$$

We conclude from Lemma 3.12 that there exist constants $C_{r,s}^{(1)} > 0$ and $C_{r,s}^{(2)} > 0$ such that, if the functions f and g belong to a single irreducible component, then

$$\begin{aligned}
|g|_{r,s;\mathcal{T}} &\leq C_{r,s}^{(1)} \frac{1+|\lambda|^{-s}}{|\lambda| \mathcal{T}^{1/3}} (1+|\nu|)^{r+s} \sum_{k=0}^{s+1} [\mathcal{T}^{-1/3} |1-\nu|]^k |f|_{0,s+1-k;\mathcal{T}} \\
&\leq C_{r,s}^{(2)} \frac{1+|\lambda|^{-s}}{|\lambda| \mathcal{T}^{1/3}} |f|_{r+s,s+1;\mathcal{T}}.
\end{aligned}$$

From the above estimate, the conclusion for r, s even integers and $m = 1$ follows.

The statement for $r, s \geq 0$ follows by interpolation. The estimate for $m \neq 1$ follows by setting $\lambda = \lambda m$.

The uniqueness of the solution holds, because if $g, h \in H$ are solutions of the equation (33), then in Fourier transform the following identity holds in $L_v^2(\mathbb{R})$, hence almost everywhere,

$$-i(\xi + \lambda)(\hat{g} - \hat{h})(\xi) = 0$$

Since $\xi + \lambda \neq 0$ almost everywhere, it follows that $\hat{g}(\xi) = \hat{h}(\xi)$ almost everywhere, hence $g = h \in H$. \square

3.3. Discrete series.

3.3.1. *Invariant distributions.* Let H_μ be an irreducible unitary representation of the discrete series with $\mu = 1 - v^2$ for $v \in \mathbb{N}$. Let \mathbb{H} be the upper half-plane.

From Appendix A,

$$\|f\|_{H_\mu}^2 := \begin{cases} \int_0^\infty \int_{-\infty}^\infty |f(x+iy)|^2 y^{v-1} dx dy, & v \geq 1 \\ \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx, & v = 0. \end{cases}$$

For $f_m = e_m \otimes f \in H_{m,\mu}$, define

$$D_{m,\mu}^\lambda(f_m) := e^\lambda \int_{\mathbb{R}} f(t+i) e^{-it\lambda m} dt,$$

and observe that $D_{m,\mu}^\lambda$ induces a functional $\bar{D}_{m,\mu}^\lambda$ on H_μ satisfying

$$\bar{D}_{m,\mu}^\lambda = D_{1,\mu}^{\lambda m}.$$

When $v = 0$, the lowest weight vector for the H_μ is $u_1(z) := (z+i)^{-1}$, and integration by parts shows $\bar{D}_{m,\mu}^\lambda(u_0) \in \mathbb{C}$. Consider the horocycle flow invariant functional D_μ^+ defined by

$$D_\mu^+(f) := \lim_{z \rightarrow \infty} f(z)(z+i).$$

The formulas from Section 2.4 of [6] show the basis obtained from u_1 by repeatedly applying the operator $1/2[X - i(U+V)]$ is orthonormal. Then formula (43) of [6] shows D_μ^+ is (sharply) in $H_\mu^{-1/2}$. So for $f_m = f \otimes e_m$,

$$(59) \quad D_{m,\mu}^\lambda(f_m) = \bar{D}_{m,\mu}^\lambda(f - D_\mu^+(f)u_1) + D_\mu^+(f)\bar{D}_{m,\mu}^\lambda(u_1)$$

is defined via the above formula. Moreover, it follows as in the third case of Lemma A.3 of [23] that $D_{m,\mu}^\lambda \in H_{m,\mu}^{-1/2}$.

For $v \geq 1$, an elementary computation from Lemma A.3 of [23] gives

Lemma 3.13. *Let $v \geq 1$, and $f \in H_\mu^\infty$. Then there is a constant $C > 0$ such that for all $z \in \mathbb{H}_\mu$,*

$$|f(z)| \leq C(1 + \text{Im}(z))^{\min\{2-(v+1)/2, -1/2\}} \|f\|_3 (1 + |z|)^{-2}.$$

Hence, for all μ such that $v \geq 0$, $D_{m,\mu}^\lambda \in H_{m,\mu}^{-3}$. The following stronger result holds.

Lemma 3.14. *Let $m \in \mathbb{Z}$ and $\nu \in \mathbb{N}$. If $\lambda m \neq 0$, then*

$$D_{m,\mu}^\lambda \in \widehat{H}_{m,\mu}^{0, -(1/2+)}.$$

Moreover, if $\lambda m < 0$, then $D_{m,\mu}^\lambda = 0$. For $\lambda m = 0$, we have two cases:

$$\begin{cases} D_{m,\mu}^\lambda \text{ is undefined,} & \text{if } \nu = 0 \\ D_{m,\mu}^\lambda = 0, & \text{if } \nu > 0. \end{cases}$$

The proof will be as in Lemma 3.3, once we have a description in Fourier transform of the upper half-plane model. For each $x + iy := z \in \mathbb{H}$, define

$$\hat{f}^\nu(\xi) := \int_{\mathbb{R}} f(z) e^{-i\xi z} dx.$$

Notice that $\bar{D}_{m,\mu}^\lambda(f) = \hat{f}^1(m\lambda)$. By Lemma 3.13 and a computation as in (59), the function $\hat{f}^\nu(\xi)$ is defined for $\xi \in \mathbb{R}^*$. By Cauchy's theorem, we get

Lemma 3.15. *Let $\xi \in \mathbb{R}$ and $y_1, y_2 > 0$. Let $f \in H_\mu^\infty$. Then $\hat{f}^{y_1}(\xi) = \hat{f}^{y_2}(\xi)$. If $\xi < 0$, then $\hat{f}^{y_1}(\xi) = 0$, and if $\nu \geq 1$, then $\hat{f}^{\nu_1}(0) = 0$.*

With this in mind, we define the Fourier transform of f to be

$$\hat{f} := \hat{f}^1.$$

Lemma 3.16. *Let $\nu \in \mathbb{Z}^+$ and $f \in H_\mu^\infty$. Then for all $z \in \mathbb{H}$,*

$$f(z) = \frac{1}{2\pi} \int_{\mathbb{R}^+} \hat{f}(\xi) e^{i\xi z} d\xi.$$

Setting $(-1)! := 1$, we get for any $\nu \in \mathbb{N}$,

$$\|f\|_0^2 = \frac{(\nu-1)!}{\pi 2^{\nu+1}} \int_{\mathbb{R}^+} |\hat{f}(\xi)|^2 \frac{d\xi}{\xi^\nu}.$$

We leave the proof of Lemma 3.16 to Appendix B. There is also a formula for Sobolev norms of functions in Fourier transform.

Lemma 3.17. *Let $s \geq 0$. Setting $(-1)! := 1$, we have for any $\nu \in \mathbb{N}$,*

$$|f|_{r,s;\mathcal{T}}^2 = \frac{(\nu-1)!}{\pi 2^{\nu+1}} (1+\mu^2)^{\frac{\nu}{2}} \int_{\mathbb{R}^+} |[I + \mu_{\mathcal{T}}^2 + (m^2 I - \hat{X}^2 - \hat{V}^2)^2]^{\frac{s}{4}} \hat{f}(\xi)|^2 \frac{d\xi}{\xi^\nu}.$$

Proof. The usual formulas

$$(60) \quad \hat{X} := (1-\nu) + 2\xi \frac{\partial}{\partial \xi}, \quad \hat{V} := -i \left((1-\nu) \frac{\partial}{\partial \xi} + \xi \frac{\partial^2}{\partial \xi^2} \right).$$

are verified on test functions $g \in H^\infty$ that satisfy $\hat{g} \in C_0^\infty(\mathbb{R}^+)$. This set is dense in H by Lemma 3.16. Thus, the identity holds. \square

Proof of Lemma 3.14. If $\lambda m < 0$, then $\bar{D}_{m,\mu}^\lambda = 0$ by Lemma 3.15, which implies $D_{m,\mu}^\lambda = 0$. Similarly, $D_{m,\mu}^\lambda = 0$ when $\lambda m = 0$ and $\nu \geq 1$. If $\lambda m = 0$ and $\nu = 0$, then $D_{m,\mu}^\lambda$ is not defined on the vector $u_1(z) = (z+i)^{-1}$, so $D_{m,\mu}^\lambda$ is not defined. The regularity statement follows as in Lemma 3.3. \square

3.3.2. *Twisted cohomological equations.* For every $\lambda \in \mathbb{R}^*$ we study the solution g to the twisted cohomological equation

$$(61) \quad \mathcal{T}(U + \lambda K)g = f$$

in every irreducible, unitary representation subspace of the foliated Sobolev space $\widehat{W}_{\mathcal{T}}^s(M \times \mathbb{T})$ of the discrete series or mock discrete series.

Theorem 3.18. *For every $r, s \geq 0$, there is a constant $C_{r,s} > 0$ such that for any irreducible unitary representation $H := H_{m,\mu}$ in the discrete series with $m \neq 0$ and for any function $f_m \in \widehat{H}^\infty \cap \text{Ann}(D_{m,\mu}^\lambda)$, there is a unique solution $g_m \in H$ satisfying (31), and moreover, for all $\mathcal{T} \geq 1$,*

$$|g_m|_{r,s;\mathcal{T}} \leq \frac{C_{r,s}}{\mathcal{T}^{1/3}} \frac{1 + |\lambda m|^{-s}}{|\lambda m|} |f_m|_{r+3s,s+1;\mathcal{T}}.$$

As in the proof of Theorem 3.4, we proceed formally and note that $f_m = f \otimes e_m$ and $g_m = g \otimes e_m$ are simple tensors. By Lemma 3.15, \hat{f} and \hat{g} are functions supported on \mathbb{R}^+ . As in the derivation of the formulas \hat{X} and \hat{V} in (60), we use Lemma 3.16 again to see that \hat{U} is multiplication by $i\xi$.

Then we may restrict our considerations of the cohomological equation (61) to

$$(62) \quad \hat{g}(\xi) := -i \frac{\hat{f}(\xi)}{\lambda(\xi + 1)},$$

by the same argument used in the proof of Theorem 3.4.

Lemma 3.19. *Theorem 3.18 is true when the representation $H_{m,\mu}$ is a mock discrete representation.*

Proof. By Lemma 3.16, $\widehat{H}_{m,\mu}$ consists of square integrable functions supported on \mathbb{R}^+ , and the measure is Lebesgue. Because the formulas for \hat{X} and \hat{V} are the same, the lemma follows identically as in the proof of Theorem 3.4. \square

In what follows, we only consider discrete series representations where $\nu \geq 1$. As above, we separately estimate \hat{g} near -1 and away from -1 .

Notice that the formulas for the vector fields \hat{X} and \hat{V} given in (60) are identical to those given for the principal and complementary series. As in Lemma 3.5,

$$\hat{g}(\xi) = -\frac{i}{\lambda} \int_0^1 \hat{f}'(-1 + t(\xi + 1)) dt.$$

Let $I = [-\frac{3}{2}, -\frac{1}{2}]$ and $L_\nu^2(I)$ be defined as in formulas (37) and (38).

Lemma 3.20. *Let $\mu \leq 0$. For every $\alpha, \beta \in \mathbb{N}$, there exists a constant $C'_{\alpha,\beta} > 0$ such that*

$$\|\hat{X}^\alpha \hat{V}^\beta \hat{g}\|_{L_\nu^2(\mathbb{R} \setminus I)} \leq \frac{C'_{\alpha,\beta}}{|\lambda|} \sum_{\substack{i+j+k \leq \alpha+\beta \\ j \leq \beta}} |1 - |\nu||^i \|X^j V^k f\|_0.$$

Proof. The proof is identical to that of Lemma 3.8. \square

Then it remains to prove

Lemma 3.21. *Let $\mu \leq 0$. Then for any $\alpha, \beta \geq 0$, there is a constant $C_{\alpha, \beta}^{(4)} > 0$ such that for all $\hat{f}, \hat{g} \in \hat{H}_\mu^\infty$ satisfying (62),*

$$\|\hat{V}^\beta \hat{X}^\alpha \hat{g}\|_{L_v^2(I)} \leq \frac{C_{\alpha, \beta}^{(4)}}{|\lambda|} (1 + |\nu|)^{3\beta} \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ j \leq \beta}} |1 - \nu|^i \|V^j X^k f\|_0.$$

This is not immediate from the proof of Lemma 3.10, because the factor $2^{\operatorname{Re} \nu}$ in formula (46) can be arbitrarily large. For $\nu \leq 1 + 2\beta$, $2^{\operatorname{Re} \nu}$ is bounded by a constant depending on β , and the proof of Lemma 3.10 holds, so Lemma 3.21 follows for this case. For $\nu > 1 + 2\beta$, we will move the problem to the setting of the principal series where $\operatorname{Re} \nu = 0$. Define

$$(63) \quad \mathcal{A} : \hat{H}_\mu^\infty \rightarrow L^2(\mathbb{R}^+) : \hat{f} \rightarrow \frac{\hat{f}(\xi)}{\xi^{\nu/2}}.$$

Notice also that \mathcal{A} is invertible, where $\mathcal{A}^{-1} : \hat{f} \rightarrow \xi^{\nu/2} \hat{f}$.

As a first step, we have

Lemma 3.22. *Let $\mu \leq 0$. Then for any $\alpha, \beta \in \mathbb{N}$, there is a constant $C_{\alpha, \beta}^{(5)} > 0$ such that for all $\hat{f}, \hat{g} \in \hat{H}_\mu^\infty$ satisfying (62),*

$$\|\hat{V}^\beta \hat{X}^\alpha \mathcal{A} \hat{g}\|_{L^2(I)} \leq \frac{C_{\alpha, \beta}^{(5)}}{|\lambda|} (1 + |\nu|)^\beta \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ j \leq \beta}} (1 + |\nu|)^i \|\hat{V}^j \hat{X}^k \mathcal{A} \hat{f}\|_{L^2(I)}.$$

Proof. Formula (55) in the case $\operatorname{Re} \nu = 0$ gives a constant $K_{\alpha, \beta} > 0$ such that

$$\begin{aligned} \|\hat{V}^\beta \hat{X}^\alpha \mathcal{A} \hat{g}\|_{L^2(I)} &\leq \frac{K_{\alpha, \beta}}{|\lambda|} (1 + |\nu|)^\beta \\ &\times \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ i \leq \beta}} \|\hat{V}^i \hat{X}^j (\hat{X} - (1 - \nu))^k \mathcal{A} \hat{f}\|_{L^2(I)} \\ &\leq \frac{C_{\alpha, \beta}''}{|\lambda|} (1 + |\nu|)^\beta \sum_{\substack{i+j+k \leq \alpha+\beta+1 \\ j \leq \beta}} (1 + |\nu|)^i \|\hat{V}^j \hat{X}^k \mathcal{A} \hat{f}\|_{L^2(I)}. \end{aligned}$$

□

Lemma 3.21 will then be obtained by estimating the norm of the linear operators \mathcal{A} and \mathcal{A}^{-1} on foliated Sobolev spaces. For any $\alpha \in \mathbb{Z}^+$, formula (39) gives universal coefficients $(a_w^{(\alpha)})$ such that

$$(64) \quad \hat{X}^\alpha (\mathcal{A} \hat{f}) = \xi^{-\nu/2} \sum_{w=0}^{\alpha} a_w^{(\alpha)} (1 - 2\nu)^w (\hat{X} - (1 - \nu))^{\alpha-w} \hat{f}.$$

$$(65) \quad \hat{X}^\alpha (\mathcal{A}^{-1} \hat{f}) = \xi^{\nu/2} \sum_{w=0}^{\alpha} a_w^{(\alpha)} (\hat{X} - (1 - \nu))^{\alpha-w} \hat{f}.$$

As in formula (54), for any integer $0 < k < \nu/2$ we have

$$\begin{aligned}\hat{V}^k\left(\frac{1}{\xi^{\nu/2}}\right) &= (-i)^k \prod_{j=0}^{k-1} (\nu/2 + j)(3\nu/2 + j) \xi^{-(\nu/2+k)} \\ \hat{V}^k(\xi^{\nu/2}) &= i^k \prod_{j=0}^{k-1} [(\nu/2)^2 - j^2] \xi^{\nu/2-k}.\end{aligned}$$

With this and Lemma 3.7, we get, for any integer $0 \leq \beta < \nu/2$, universal coefficients $(b_{l,j,k,m}^{(\beta)'})$ such that

$$\begin{aligned}\hat{V}^\beta(\mathcal{A}\hat{f}) &= \sum_{\substack{l+j+m \leq \beta \\ k \leq m \\ j \leq \beta}} b_{l,j,k,m}^{(\beta)'} \prod_{\tilde{l}=0}^{l-1} (\nu/2 + \tilde{l})(3\nu/2 + \tilde{l}) \prod_{\tilde{m}=0}^{m-1} (\nu/2 + \tilde{m}) \\ &\quad \times \xi^{-(\nu/2+l+m)} (\hat{X} - (1-\nu))^k \hat{V}^j \hat{f};\end{aligned}\tag{66}$$

$$\begin{aligned}\hat{V}^\beta(\mathcal{A}^{-1}\hat{f}) &= \sum_{\substack{l+j+m \leq \beta \\ k \leq m \\ j \leq \beta}} b_{l,j,k,m}^{(\beta)' } (-1)^l \prod_{\tilde{l}=0}^{l-1} [(\nu/2)^2 - \tilde{l}^2] \prod_{\tilde{m}=0}^{m-1} (\nu/2 + \tilde{m}) \\ &\quad \times \xi^{\nu/2-l-m} (\hat{X} - (1-\nu))^k \hat{V}^j \hat{f}.\end{aligned}\tag{67}$$

Lemma 3.23. *Let $f \in \hat{H}_\mu^\infty$. Then for any $\alpha \in \mathbb{N}$ and integer $0 \leq \beta < \frac{\nu}{2}$, there is a constant $C_{\alpha,\beta}^{(6)} > 0$ such that*

$$\|\hat{V}^\beta \hat{X}^\alpha(\mathcal{A}\hat{f})\|_{L^2(I)} \leq C_{\alpha,\beta}^{(6)} \sum_{\substack{i+j+k \leq \alpha+2\beta \\ j+k \leq \alpha+\beta \\ j \leq \beta}} (1+|\nu|)^i \|\hat{V}^j \hat{X}^k \hat{f}\|_{L_\nu^2(I)}.$$

Proof. By (64) and (66), we get universal coefficients $(b_{w,l,j,k,m}^{(\alpha,\beta)})$ such that

$$\begin{aligned}\hat{V}^\beta \hat{X}^\alpha(\mathcal{A}\hat{f}) &= \hat{V}^\beta \left(\sum_{w=0}^{\alpha} a_w^{(\alpha)} (1-2\nu)^w \xi^{-\nu/2} \cdot (\hat{X} - (1+\nu))^{\alpha-w}(\hat{f}) \right) \\ &= \sum_{w=0}^{\alpha} a_w^{(\alpha)} (1-2\nu)^w \cdot \hat{V}^\beta \left(\xi^{-\nu/2} \cdot (\hat{X} - (1+\nu))^{\alpha-w}(\hat{f}) \right) \\ &= \sum_{w=0}^{\alpha} \sum_{\substack{l+j+m \leq \beta \\ k \leq m}} b_{w,l,j,k,m}^{(\alpha,\beta)} (1-2\nu)^w \prod_{\tilde{l}=0}^{l-1} (\nu/2 + \tilde{l})(3\nu/2 + \tilde{l}) \prod_{\tilde{m}=0}^{m-1} (\nu/2 + \tilde{m}) \\ &\quad \times \left(\frac{1}{\xi^{l+m+\nu/2}} \right) (\hat{X} - (1-\nu))^k \hat{V}^j (\hat{X} - (1-\nu))^{\alpha-w}(\hat{f})\end{aligned}$$

By the commutation relation $[\hat{X}, \hat{V}] = -2\hat{V}$, it follows that there are constants $C_{\alpha, \beta} > 0$ such that

$$\begin{aligned}
\|\hat{V}^\beta \hat{X}^\alpha (\mathcal{A}\hat{f})\|_{L^2(I)} &\leq C_{\alpha, \beta} \sum_{w=0}^{\alpha} \sum_{\substack{l+j+m \leq \beta \\ k \leq m}} (1+|\nu|)^{w+2l+m} \\
&\quad \times \left\| \frac{1}{\xi^{l+m+\nu/2}} (\hat{X} - (1-\nu))^k \hat{V}^j (\hat{X} - (1-\nu))^{\alpha-w} \hat{f} \right\|_{L^2(I)} \\
&\leq C_{\alpha, \beta} \sum_{w=0}^{\alpha} \sum_{\substack{l+j+m \leq \beta \\ k \leq m}} (1+|\nu|)^{w+2l+m} \\
&\quad \times \sum_{\tilde{k} \leq k} \|\hat{V}^j (\hat{X} - (1-\nu))^{\alpha+\tilde{k}-w} \hat{f}\|_{L^2_\nu(I)} \\
&\leq C_{\alpha, \beta} \sum_{\substack{i+j+k \leq \alpha+2\beta \\ j+k \leq \alpha+\beta \\ j \leq \beta}} (1+|\nu|)^i \|\hat{V}^j (\hat{X} - (1-\nu))^k \hat{f}\|_{L^2_\nu(I)} \\
&\leq C_{\alpha, \beta} \sum_{\substack{i+j+k \leq \alpha+2\beta \\ j+k \leq \alpha+\beta \\ j \leq \beta}} (1+|\nu|)^i \|\hat{V}^j \hat{X}^k \hat{f}\|_{L^2_\nu(I)}.
\end{aligned}$$

□

Similarly, we have

Lemma 3.24. *Let $\mathcal{A}^{-1}\hat{h} \in \hat{H}_\mu^\infty$. Then for any $\alpha \in \mathbb{N}$ and integer $0 \leq \beta < \frac{\nu}{2}$, there is a constant $C_{\alpha, \beta}^{(7)} > 0$ such that*

$$\|\hat{V}^\beta \hat{X}^\alpha (\mathcal{A}^{-1}\hat{h})\|_{L^2_\nu(I_\lambda)} \leq C_{\alpha, \beta}^{(7)} \sum_{\substack{i+j+k \leq \alpha+2\beta \\ j+k \leq \alpha+\beta \\ j \leq \beta}} (1+|\nu|)^i \|\hat{V}^j \hat{X}^k \hat{h}\|_{L^2(I_\lambda)}.$$

Proof. By formulas (65) and (67) in place of (64) and (66), we get

$$\begin{aligned}
\hat{V}^\beta \hat{X}^\alpha (\mathcal{A}^{-1}\hat{h}) &= \sum_{w=0}^{\alpha} \sum_{\substack{j+m \leq \beta \\ k \leq m}} b_{wljkm}^{(\alpha, \beta)} (-1)^l \prod_{\tilde{l}=0}^{l-1} [(\nu/2)^2 - \tilde{l}^2] \prod_{\tilde{m}=0}^{m-1} (\nu/2 + \tilde{m}) \\
&\quad \times \xi^{\nu/2-m-l} (\hat{X} - (1-\nu))^k \hat{V}^j (\hat{X} - (1-\nu))^{\alpha-w} (\hat{h}).
\end{aligned}$$

Then as in the proof of Lemma 3.23, there are constants $C_{\alpha,\beta} > 0$ such that

$$\begin{aligned} \|\hat{V}^\beta \hat{X}^\alpha (\mathcal{A}^{-1} \hat{h})\|_{L_v^2(I_\lambda)} &\leq C_{\alpha,\beta} \sum_{w=0}^{\alpha} \sum_{\substack{j+m \leq \beta \\ k \leq m}} (1+|v|)^{w+2l+m} \\ &\quad \times \|\xi^{v/2-l-m} (\hat{X} - (1-v))^k \hat{V}^j (\hat{X} - (1-v))^{\alpha-w} \hat{h}\|_{L_v^2(I_\lambda)} \\ &\leq C_{\alpha,\beta} \sum_{\substack{i+j+k \leq \alpha+2\beta \\ j+k \leq \alpha+\beta \\ j \leq \beta}} (1+|v|)^i \|\hat{V}^j (\hat{X} - (1-v))^k \hat{f}\|_{L^2(I_\lambda)}. \end{aligned}$$

□

Proof of Lemma 3.21. Let $h := \mathcal{A}g$, and observe from (62) that

$$\hat{h}(\xi) = -i \frac{\mathcal{A}\hat{f}(\xi)}{\lambda(\xi+1)}$$

Then by Lemma 3.24, Lemma 3.22 and Lemma 3.23, we get

$$\begin{aligned} \|\hat{V}^\beta \hat{X}^\alpha \hat{g}\|_{L_v^2(I_\lambda)} &= \|\hat{V}^\beta \hat{X}^\alpha \mathcal{A}^{-1} \hat{h}\|_{L_v^2(I_\lambda)} \leq C_{\alpha,\beta}^{(7)} \\ &\quad \times \sum_{\substack{i+j+k \leq \alpha+2\beta \\ j+k \leq \alpha+\beta \\ j \leq \beta}} (1+|v|)^i \|\hat{V}^j \hat{X}^k \hat{h}\|_{L^2(I_\lambda)} \\ &\leq \frac{C_{\alpha,\beta}^{(7)} C_{\alpha,\beta}^{(5)}}{|\lambda|} \sum_{\substack{i+j+k \leq \alpha+2\beta \\ j+k \leq \alpha+\beta \\ j \leq \beta}} (1+|v|)^i \\ &\quad \times \sum_{\substack{i'+j'+k' \leq j+k+1 \\ j' \leq j}} (1+|v|)^{i'} \|\hat{V}^{j'} \hat{X}^{k'} \mathcal{A}\hat{f}\|_{L_v^2(I_\lambda)} \\ &\leq \frac{C_{\alpha,\beta}^{(7)} C_{\alpha,\beta}^{(5)} C_{\alpha,\beta}^{(6)}}{|\lambda|} (1+|v|)^\beta \sum_{\substack{i+j+k \leq \alpha+2\beta \\ j+k \leq \alpha+\beta \\ j \leq \beta}} (1+|v|)^i \\ &\quad \times \sum_{\substack{i'+j'+k' \leq j+k+1 \\ j' \leq j}} (1+|v|)^{i'} \sum_{\substack{i''+j''+k'' \leq k'+2j' \\ j''+k'' \leq j'+k' \\ j'' \leq j'}} (1+|v|)^{i''} \|\hat{V}^{j''} \hat{X}^{k''} \hat{f}\|_{L_v^2(I_\lambda)}. \end{aligned}$$

In the last summation of the above formula we have

$$\begin{aligned} i+i'+i''+j''+k'' &\leq i+i'+k'+2j' \leq i+j+k+\beta+1 \leq \alpha+3\beta+1, \\ j''+k'' &\leq j'+k' \leq j+k+1 \leq \alpha+\beta+1. \end{aligned}$$

This concludes the proof of Lemma 3.21. □

For a general $\lambda \in \mathbb{R}^*$ and for rescaled Sobolev norms we have a similar statement which can be immediately derived from Lemma 3.20, Lemma 3.21 and from a calculation similar to the one in formula (57) in the proof of Lemma 3.11.

Lemma 3.25. *Let $\mu \leq 0$. For every $\alpha, \beta \geq 0$ there exists a constant $C_{\alpha,\beta}^{(4)} > 0$ such that for all $\lambda \neq 0$ and for all $\mathcal{T} \geq 1$, the solution $g \in L_V^2(\mathbb{R})$ of the rescaled equation $\mathcal{T}(U + i\lambda)g = f \in L_V^2(\mathbb{R})$ satisfies the following estimate with respect to the rescaled Sobolev norms:*

$$\begin{aligned} \|V_{\mathcal{T}}^{\beta} X_{\mathcal{T}}^{\alpha} g\|_0 &\leq \frac{C_{\alpha,\beta}^{(4)}}{|\lambda| \mathcal{T}^{1/3}} (1 + |\mathbf{v}|)^{3\beta} (1 + |\lambda|^{-\beta}) \\ &\quad \times \sum_{i+j+k \leq \alpha+\beta+1} [\mathcal{T}^{-1/3} |1 - \mathbf{v}|]^i \|V_{\mathcal{T}}^j X_{\mathcal{T}}^k f\|_0. \end{aligned}$$

Now we may prove Theorem 3.18.

Proof of Theorem 3.18. We proceed as in the proof of Theorem 3.4. We claim that, for $r, s \in \mathbb{N}$ even integers, there exist constants $C_{r,s}^{(1)} > 0$ and $C_{r,s}^{(2)} > 0$ such that, if the functions f and g belong to a single irreducible component, then

$$\begin{aligned} (68) \quad |g|_{r,s;\mathcal{T}} &\leq \frac{C_{r,s}^{(1)}}{\mathcal{T}^{1/3}} \frac{1 + |\lambda|^{-s}}{|\lambda|} (1 + |\mathbf{v}|)^{r+3s} \sum_{k=0}^{s+1} [\mathcal{T}^{-1/3} |1 - \mathbf{v}|]^k |f|_{0,s+1-k;\mathcal{T}} \\ &\leq \frac{C_{r,s}^{(2)}}{\mathcal{T}^{1/3}} \frac{1 + |\lambda|^{-s}}{|\lambda|} |f|_{r+3s,s+1;\mathcal{T}}. \end{aligned}$$

Since the Casimir operator \square takes the value $\mu = 1 - \mathbf{v}^2$ on any irreducible, unitary representation $H_{m,\mu}$, by expanding the operator $(I + \square^2)^{r/2} (I + \square^2 + \widehat{\Delta}_{\mathcal{T}}^2)^{s/2}$ into polynomial expression in $X_{\mathcal{T}}$ and $V_{\mathcal{T}}$, and by the commutation relations, we derive that there exists a constant $C_{r,s}^{(3)} > 0$ such that

$$(69) \quad |g|_{r,s;\mathcal{T}} \leq C_{r,s}^{(3)} (1 + \mu^2)^{r/4} \sum_{k+m+\alpha+\beta \leq s} [\mathcal{T}^{-1/3} |1 - \mathbf{v}|]^k \|K^m V_{\mathcal{T}}^{\beta} X_{\mathcal{T}}^{\alpha} g\|_0.$$

By the above bound on the norms the estimate in formula (68) follows directly from Lemma 3.25. From the estimate (68), the conclusion for r, s even integers and $m = 1$ follows. The statement for $r, s \geq 0$ follows by interpolation. As in the proof of Theorem 3.4, the estimate for $m \neq 1$ follows by setting $\lambda = \lambda m$.

The uniqueness of the solution holds as in the proof of Theorem 3.4. \square

We can now prove Theorem 2.1 on the classification of invariant distributions and Theorem 2.2 on Sobolev bounds for solutions of the cohomological equation for the twisted horocycle flow.

Proof of Theorem 2.1. It follows from Theorem 3.4 and Theorem 3.18 that the space of invariant distributions is one dimensional. The regularity part of the statement follows from Lemma 3.3 and Lemma 3.14. \square

Proof of Theorem 2.2. The bounds with respect to the foliated Sobolev norms follows from Theorems 3.4 and 3.18 by orthogonality since the above estimates are uniform with respect to the Casimir parameter.

The bounds with respect to the Sobolev norms can be proved as follows. First let $s \in \mathbb{N}$ be even. Since the vector fields U and K commute, for all $j \in \mathbb{N}$ we have

$$(U + \lambda K)U^j g = U^j f,$$

hence the bound with respect to foliated Sobolev norms holds for the functions $U^j g$ in terms of the function $U^j f$, for all $j \in \mathbb{N}$. Then

$$\begin{aligned} \|g\|_s &\leq \sum_{j=0}^s |U^j g|_{0,s-j} \\ &\leq \frac{C_s}{|\lambda|} \sum_{j=0}^s (1 + |\lambda|^{-(s-j)}) |U^j f|_{3(s-j),s-j+1} \\ &\leq \frac{C_s}{|\lambda|} (1 + |\lambda|^{-s}) \|f\|_{4s+1}. \end{aligned}$$

The estimates for general Sobolev norms follows by interpolation.

Finally, the solution is unique in $L^2(M \times \mathbb{T})$ by Theorem 3.4 and Theorem 3.18

□

4. SCALING OF INVARIANT DISTRIBUTIONS

In this section we prove estimates on the scaled foliated Sobolev norms of invariant distributions for the twisted cohomological equation.

Let us consider an irreducible, unitary representation $H := H_{m,\mu}$ of the group $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{T}$ and let $\lambda m \neq 0$. For $\mathcal{T} > \mathcal{T}'$, we want to estimate

$$|D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}} := \sup_{F \in \hat{H}^s} \left\{ |D_{m,\mu}^\lambda(F)| : |F|_{r,s;\mathcal{T}} = 1 \right\}$$

in terms of $|D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}'} := \sup_F \left\{ |D_{m,\mu}^\lambda(F)| : |F|_{r,s;\mathcal{T}'} = 1 \right\}$. We introduce the following foliated Sobolev Lyapunov norms on \hat{H}^s . For all $\mathcal{T} \geq 1$,

$$|F|_{r,s;\mathcal{T}}^\mathcal{L} := \sup_{\tau > \mathcal{T}} \left(\frac{\tau}{\mathcal{T}} \right)^{1/6} |F|_{r,s;\tau}.$$

From definitions, the following holds.

Lemma 4.1. *Let $r, s \geq 0$. For all $\mathcal{T} > \mathcal{T}' \geq 1$ and for all $F \in H^s$,*

$$\begin{aligned} |F|_{r,s;\mathcal{T}} &\leq |F|_{r,s;\mathcal{T}'}^\mathcal{L}; \\ |F|_{r,s;\mathcal{T}}^\mathcal{L} &\leq \left(\frac{\mathcal{T}'}{\mathcal{T}} \right)^{1/6} |F|_{r,s;\mathcal{T}'}^\mathcal{L}. \end{aligned}$$

Lemma 4.1 immediately gives

Corollary 4.2. *Let $r, s > 1/2$ and $\mathcal{T} > \mathcal{T}' > 1$. Then*

$$\begin{aligned} |D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}}^\mathcal{L} &\leq |D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}'}^\mathcal{L}, \\ |D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}'}^\mathcal{L} &\leq \left(\frac{\mathcal{T}'}{\mathcal{T}}\right)^{1/6} |D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}}^\mathcal{L}. \end{aligned}$$

Our strategy is to prove comparison bounds between the foliated Sobolev dual norms and the foliated Sobolev Lyapunov dual norms of the invariant distribution in every irreducible, unitary representation.

Hence, it remains to prove a bound from above for the foliated Sobolev norm $|D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}}^\mathcal{L}$ in terms of the foliated Sobolev Lyapunov norm $|D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}'}^\mathcal{L}$. We consider the principal and complementary series together, while the discrete series is handled separately.

4.1. Principal and complementary series. Throughout this subsection, given an integer $m \in \mathbb{Z}$ and a Casimir parameter $\mu > 0$, we let $H := H_{m,\mu}$ be an irreducible, unitary representation of the principal or complementary series for the group $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{T}$. We prove the following theorem.

Theorem 4.3. *For every $r \geq 0$ and $s > 1/2$ there is a constant $C_{r,s} > 0$ such that for all $\mathcal{T} \geq \mathcal{T}' \geq 1$ and $\lambda \in \mathbb{R}$ such that $\lambda m \neq 0$, the distribution $D_{m,\mu}^\lambda \in \widehat{H}^{-r,-s}$ satisfies the scaling estimates*

$$|D_{m,\mu}^\lambda|_{-(r+s),-s;\mathcal{T}'}^\mathcal{L} \leq C_{r,s} \left(\frac{\mathcal{T}'}{\mathcal{T}}\right)^{1/6} (1 + |\lambda m|^{-2s}) |D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}}^\mathcal{L}.$$

By Corollary 4.2, it is enough to prove the following proposition.

Proposition 4.4. *For every $r \geq 0$ and $s > 1/2$ there is a constant $C_{r,s} > 0$ such that for all $\mathcal{T} \geq 1$ and $\lambda \in \mathbb{R}$ such that $\lambda m \neq 0$, the distribution $D_{m,\mu}^\lambda \in \widehat{H}^{-r,-s}$ satisfies*

$$|D_{m,\mu}^\lambda|_{-(r+s),-s;\mathcal{T}}^\mathcal{L} \leq C_{r,s} (1 + |\lambda m|^{-2s}) |D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}}^\mathcal{L}.$$

Once more the general case can be derived from the case $m = 1$. We will therefore restrict our argument to that case and prove the statement for the distributions $D_{1,\mu}^\lambda$ whenever $\lambda \neq 0$. We again let

$$I_\lambda := [\lambda - |\lambda|/2, \lambda + |\lambda|/2].$$

Now let us consider, for all $\tau \geq 1$, the operator U_τ formally defined on H in Fourier transform as follows:

$$(70) \quad \hat{U}_\tau(\hat{f})(\xi) = \tau^{1/6} \hat{f}(\lambda + \tau^{1/3}(\xi - \lambda)), \quad \text{for all } f \in H.$$

In fact, it can be proved that the following bounds hold:

Lemma 4.5. *For all $\tau \geq 1$ and $\hat{f} \in C_0^\infty(I_\lambda)$,*

$$\frac{1}{\sqrt{3}} \|f\|_0 \leq \|U_\tau f\|_0 \leq \sqrt{3} \|f\|_0.$$

Lemma 4.5 is proved in Appendix B.

We recall from (29) that the Fourier transforms of the X and V operators are

$$\hat{X} := (1 - \nu) + 2\xi \frac{\partial}{\partial \xi}, \quad \hat{V} := -i \left((1 - \nu) \frac{\partial}{\partial \xi} + \xi \frac{\partial^2}{\partial \xi^2} \right).$$

Lemma 4.6. *The following formulas hold for all $\tau > 1$:*

$$\begin{aligned} \hat{U}_\tau^{-1} \hat{X} \hat{U}_\tau &= \hat{X} - 2\lambda \tau^{1/3} (1 - \tau^{-1/3}) \frac{\partial}{\partial \xi}; \\ \hat{U}_\tau^{-1} \hat{V} \hat{U}_\tau &= \tau^{-1/3} \hat{V} + i\lambda \tau^{2/3} (1 - \tau^{-1/3}) \frac{\partial^2}{\partial \xi^2}. \end{aligned}$$

We have the following scaling estimate.

Lemma 4.7. *For every $r, s \geq 0$, there exists a constant $C'_{r,s} > 0$ such that for all $\lambda \neq 0$, for all $\tau \geq \mathcal{T} \geq 1$ and for all $\hat{f} \in C_0^\infty(I_\lambda)$ the following bound holds:*

$$|U_{\tau/\mathcal{T}} f|_{r,s;\tau} \leq C'_{r,s} (1 + |\lambda|^{-s}) |f|_{r+s,\mathcal{T}}.$$

Proof. By Lemma 4.6, we have

$$\begin{aligned} \hat{U}_{\tau/\mathcal{T}}^{-1} \hat{X} \hat{U}_{\tau/\mathcal{T}} &= \left(\frac{\tau}{\mathcal{T}}\right)^{-1/3} \hat{X}_\mathcal{T} - 2[1 - (\frac{\tau}{\mathcal{T}})^{-1/3}] \mathcal{T}^{-1/3} \lambda \frac{\partial}{\partial \xi}; \\ \hat{U}_{\tau/\mathcal{T}}^{-1} \hat{V} \hat{U}_{\tau/\mathcal{T}} &= \left(\frac{\tau}{\mathcal{T}}\right)^{-1/3} \hat{V}_\mathcal{T} + i[1 - (\frac{\tau}{\mathcal{T}})^{-1/3}] \mathcal{T}^{-2/3} \lambda \frac{\partial^2}{\partial \xi^2}. \end{aligned} \tag{71}$$

We then observe that

$$\begin{aligned} \mathcal{T}^{-1/3} \lambda \frac{\partial}{\partial \xi} &= \frac{\lambda}{\xi} (\hat{X}_\mathcal{T} - \mathcal{T}^{-1/3} (1 - \nu)); \\ \mathcal{T}^{-2/3} \lambda \frac{\partial^2}{\partial \xi^2} &= \frac{\lambda}{\xi} \left(i\hat{V}_\mathcal{T} - \frac{(1 - \nu)\mathcal{T}^{-1/3}}{2\xi} (\hat{X}_\mathcal{T} - \mathcal{T}^{-1/3} (1 - \nu)) \right). \end{aligned} \tag{72}$$

We recall the identities (50) and (54). For all integers $s \in \mathbb{N}$ (by induction), we have

$$\hat{X}^s \left(\frac{1}{\xi} \right) = (-1 - \nu)^s \left(\frac{1}{\xi} \right) \quad \text{and} \quad \hat{V}^s \left(\frac{1}{\xi} \right) = (-i)^s s! \left(\prod_{j=1}^s (j + \nu) \right) \left(\frac{1}{\xi^{s+1}} \right).$$

It follows that for all $s \in \mathbb{N}$ there exists a constant $C'_s > 0$ such that for all $\hat{f} \in C_0^\infty(I_\lambda)$ we have

$$|\hat{U}_{\tau/\mathcal{T}} \hat{f}|_{r,s;\tau} \leq C'_s (1 + |\nu|)^s (1 + |\lambda|^{-s}) |\hat{f}|_{r,s;\mathcal{T}} \leq C'_s (1 + |\lambda|^{-s}) |\hat{f}|_{r+s,\mathcal{T}}.$$

The statement then follows by interpolation. \square

Lemma 4.8. *For every $r, s \geq 0$ there exists a constant $C''_{r,s} > 0$ such that the following holds. For any function $\hat{f} \in \widehat{H}^\infty$ and for any $\lambda \neq 0$, there exists a function $\hat{f}_\lambda \in C_0^\infty(I_\lambda)$ with $\hat{f}_\lambda(-\lambda) = \hat{f}(-\lambda)$ such that for any $r, s \geq 0$ and $\mathcal{T} \geq 1$ we have*

$$|f_\lambda|_{r,s;\mathcal{T}} \leq C''_{r,s} (1 + |\lambda|^{-s}) |f|_{r,s;\mathcal{T}}.$$

Proof. Let $\hat{\phi} \in C_0^\infty(-1/2, 1/2)$ be any function such that $\hat{\phi}(0) = 1$. We let

$$\hat{\phi}_\lambda(\xi) := \hat{\phi}\left(\frac{\xi + \lambda}{|\lambda|}\right), \quad \text{for all } \xi \in \mathbb{R}.$$

By construction we have that the function $\hat{\phi}_\lambda \in C_0^\infty(I_\lambda)$. By an induction argument based on the formulas (29) for the Fourier transforms \hat{X} , \hat{V} of the operators X , V , we derive the following bounds. For every $\alpha, \beta \in \mathbb{N}$ there exists a constant $C''_{\alpha,\beta} > 0$ such that

$$\|\hat{V}_\tau^\beta \hat{X}_\tau^\alpha \hat{\phi}_\lambda\|_0 \leq C''_{\alpha,\beta} (1 + |\lambda|^{-\beta}) (1 + \tau^{-1/3} |1 - \nu|)^{\alpha+\beta}.$$

Let then $\hat{f}_\lambda := \phi_\lambda \hat{f}$. By construction we immediately have that $\hat{f}_\lambda \in C_0^\infty(I_\lambda)$ and $\hat{f}_\lambda(-\lambda) = \hat{f}(-\lambda)$. Finally from the Leibniz-type formula (39) and from that of Lemma 3.7 we derive that for all $r \geq 0$ and all integer $s \in \mathbb{N}$ there exists a constant $C''_{r,s} > 0$ such that

$$|\hat{f}_\lambda|_{r,s;\mathcal{T}} \leq C''_{r,s} (1 + |\lambda|^{-s}) |f|_{r,s;\mathcal{T}}.$$

The estimate in the statement is thus proved for integer exponents and follows by interpolation in the general case. \square

Proof of Proposition 4.4. For simplicity of notation, we again let $D^\lambda := D_{1,\mu}^\lambda$. Let $\hat{f}_\lambda \in C_0^\infty(I_\lambda)$ be the function constructed in Lemma 4.8. By definition we have

$$D^\lambda(f) = D^\lambda(f_\lambda) = \left(\frac{\tau}{\mathcal{T}}\right)^{-1/6} D^\lambda(U_{\tau/\mathcal{T}} f_\lambda).$$

By Lemma 4.7 and Lemma 4.8, it follows that, for all $\tau \geq \mathcal{T} \geq 1$, we have

$$\begin{aligned} |D^\lambda(f)| &\leq \left(\frac{\tau}{\mathcal{T}}\right)^{-1/6} |D^\lambda|_{-r,-s;\tau} |U_{\tau/\mathcal{T}} f_\lambda|_{r,s;\tau} \\ (73) \quad &\leq C'_{r,s} \left(\frac{\tau}{\mathcal{T}}\right)^{-1/6} |D^\lambda|_{-r,-s;\tau} (1 + |\lambda|^{-s}) |f_\lambda|_{r+s,s;\mathcal{T}} \\ &\leq C'_{r,s} C''_{r,s} \left(\frac{\tau}{\mathcal{T}}\right)^{-1/6} |D^\lambda|_{-r,-s;\tau} (1 + |\lambda|^{-2s}) |f|_{r+s,s;\mathcal{T}}. \end{aligned}$$

Hence, by definition

$$\begin{aligned} |D^\lambda|_{-(r+s),-s;\mathcal{T}} &\leq C'_{r,s} C''_{r,s} (1 + |\lambda|^{-2s}) \inf_{\tau \geq \mathcal{T}} \left(\frac{\tau}{\mathcal{T}}\right)^{-1/6} |D^\lambda|_{-r,-s;\tau} \\ (74) \quad &\leq C'_{r,s} C''_{r,s} (1 + |\lambda|^{-2s}) |D^\lambda|_{-r,-s;\mathcal{T}}^{\mathcal{L}}. \end{aligned}$$

\square

As remarked above, Theorem 4.3 immediately follows from Corollary 4.2 and Proposition 4.4, hence its proof is complete.

4.2. Discrete series. Let \mathbb{H} be the upper half-plane model for a holomorphic discrete series or mock discrete series irreducible, unitary representation of $\mathrm{SL}(2, \mathbb{R})$ with Casimir parameter $\mu := 1 - \nu^2$, where $\nu \geq 0$ is an integer, and let $m \in \mathbb{Z} \setminus \{0\}$. We prove the following theorem.

Theorem 4.9. *Let $r \geq 0$, $s > 1/2$ and $\lambda \in \mathbb{R}^*$. Then there is a constant $C_{r,s} > 0$ such that for all $\mathcal{T} \geq \mathcal{T}' \geq 1$, $D_{m,\mu}^\lambda \in \widehat{H}^{-r,-s}$ satisfies*

$$|D_{m,\mu}^\lambda|_{-(r+2s),-s;\mathcal{T}'} \leq C_{r,s} \left(\frac{\mathcal{T}'}{\mathcal{T}}\right)^{1/6} (1 + |\lambda m|^{-3s}) |D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}}.$$

Proof. Notice that by Lemma 3.15, $D_{m,\mu}^\lambda = 0$ when $\lambda m < 0$. Hence, we may assume $\lambda m > 0$. By Corollary 4.2 again, it is enough to prove the following proposition.

Proposition 4.10. *Let $r \geq 0$, $s > 1/2$, and $\lambda \in \mathbb{R}^*$ with $\lambda m > 0$. There is a constant $C_{r,s} > 0$ such that for all $\mathcal{T} \geq 1$, $D_{m,\mu}^\lambda \in \widehat{H}^{-r,-s}$ satisfies*

$$|D_{m,\mu}^\lambda|_{-(r+2s),-s;\mathcal{T}} \leq C_{r,s} (1 + |\lambda m|^{-3s}) |D_{m,\mu}^\lambda|_{-r,-s;\mathcal{T}}^{\mathcal{L}}.$$

The general case can again be derived from the case $\lambda \in \mathbb{R}^*$ and $m = 1$. Define the dilation operator U_τ as in (70). We cannot immediately conclude Proposition 4.10 from the proof of Proposition 4.4, because Lemma 4.5 does not hold for all λ and ν . Instead, we have

Lemma 4.11. *Assume $\lambda \geq \nu$, then there is a constant $C > 1$ such that for all $\tau \geq 1$ and $\hat{f} \in C_0^\infty(I_\lambda)$,*

$$\frac{1}{C} \|f\|_0 \leq \|U_\tau f\|_0 \leq C \|f\|_0.$$

The proof is left for Appendix B.

Proof of Proposition 4.10. If $\lambda \geq \nu + 1$, then by Lemma 4.11, Proposition 4.10 follows as in Proposition 4.4.

Now assume $\lambda < \nu + 1$. To simplify notation, for any $\alpha \in \mathbb{R}^*$, set $D^\alpha := D_{1,\mu}^\alpha$. Because $\lambda > 0$, let $\kappa \in \mathbb{R}^+$ satisfy $\lambda e^\kappa = \nu + 1$. Observe that for any $f \in \widehat{H}^\infty$,

$$D^\lambda(f) = (a_{-\kappa} D^\lambda)(f \circ a_{-\kappa}).$$

Moreover,

$$(U + i(\nu + 1))(a_{-\kappa} D^\lambda) = 0.$$

Because the space of invariant distributions in $\mathcal{E}'(H)$ is one-dimensional, we have $a_{-\kappa} D^\lambda \in \langle D^{\nu+1} \rangle$, where for any $h \in \widehat{H}^\infty$, and $D^{\nu+1}$ is defined as usual by

$$D^{\nu+1}(h) = e^{\nu+1} \int_{\mathbb{R}} h(t) e^{-i(\nu+1)t} dt.$$

Observe that Lemma 4.7 and Lemma 4.8 also holds for the discrete series. Let $f_{\nu+1} \in H^\infty$ correspond to $f \circ a_{-\kappa}$ as in Lemma 4.8. Then

$$\begin{aligned} |D^\lambda(f)| &= |(a_{-\kappa} D^\lambda)(f_{\nu+1})| \\ (75) \quad &= \left(\frac{\tau}{\mathcal{T}}\right)^{-1/6} |(a_{-\kappa} D^\lambda)(U_{\tau/\mathcal{T}}(f_{\nu+1}))|. \end{aligned}$$

By Lemma 4.7 and Lemma 4.8, there exists a constant $C_{r,s}^{(3)} := C'_{r,s} C''_{r,s} > 0$ such that

$$\begin{aligned}
 |D^\lambda(f)| &\leq \left(\frac{\tau}{\mathcal{J}}\right)^{-1/6} |a_{-\kappa} D^\lambda|_{-r,-s;\tau} |U_{\tau/\mathcal{J}}(f_{v+1})|_{r,s;\tau} \\
 &\leq C'_{r,s} \left(\frac{\tau}{\mathcal{J}}\right)^{-1/6} (1 + |\lambda|^{-s}) |a_{-\kappa} D^\lambda|_{-r,-s;\tau} |f_{v+1}|_{r+s,s;\mathcal{J}} \\
 &\leq C_{r,s}^{(3)} \left(\frac{\tau}{\mathcal{J}}\right)^{-1/6} (1 + |\lambda|^{-2s}) |a_{-\kappa} D^\lambda|_{-r,-s;\tau} |f \circ a_{-\kappa}|_{r+s,s;\mathcal{J}} \\
 (76) \quad &\leq C_{r,s}^{(3)} \left(\frac{\tau}{\mathcal{J}}\right)^{-1/6} (1 + |\lambda|^{-2s}) |a_{-\kappa} D^\lambda|_{-r,-s;\tau} |f|_{r+s,s;\mathcal{J}},
 \end{aligned}$$

where the last inequality holds because

$$(77) \quad |V(f \circ a_{-\kappa})| = e^{-\kappa} |(Vf) \circ a_{-\kappa}| \leq \frac{\lambda}{v+1} |(Vf) \circ a_{-\kappa}|.$$

We then estimate the norm $|a_{-\kappa} D^\lambda|_{-r,-s;\tau}$ of the invariant distribution after geodesic scaling.

Lemma 4.12. *We have*

$$|a_{-\kappa} D^\lambda|_{-r,-s;\tau} \leq \left(\frac{v+1}{\lambda}\right)^s |D^\lambda|_{-r,-s;\tau}.$$

Proof. Notice that

$$\begin{aligned}
 |a_{-\kappa} D^\lambda|_{-r,-s;\tau} &= \sup_{f \in \hat{H}^\infty} \left\{ |a_{-\kappa} D^\lambda(f)| : |f|_{r,s;\tau} = 1 \right\} \\
 &= \sup_{f \in \hat{H}^\infty} \left\{ |D^\lambda(f \circ a_{\kappa})| : |f|_{r,s;\tau} = 1 \right\} \\
 &= \sup_{f \in \hat{H}^\infty} \left\{ |D^\lambda(f)| : |f \circ a_{-\kappa}|_{r,s;\tau} = 1 \right\} \\
 &\leq \left(\frac{v+1}{\lambda}\right)^s \sup_{f \in \hat{H}^\infty} \left\{ |D^\lambda(f)| : |f|_{r,s;\tau} = 1 \right\},
 \end{aligned}$$

where the last inequality holds since

$$|Vf| = |V(f \circ a_{-\kappa} \circ a_{\kappa})| = e^{\kappa} |V(f \circ a_{-\kappa})| \leq \frac{v+1}{\lambda} |V(f \circ a_{-\kappa})|.$$

□

Therefore,

$$(76) \leq C_s^{(3)} \left(\frac{\tau}{\mathcal{J}}\right)^{-1/6} \left(\frac{v+1}{\lambda}\right)^s (1 + |\lambda|^{-2s}) |D^\lambda|_{-s;\tau} |f|_{r+s,s;\mathcal{J}}.$$

Proposition 4.10 now follows as in (74).

□

This completes the proof of Theorem 4.9.

□

5. SOBOLEV TRACE THEOREM

In this section we prove a Sobolev trace theorem for horocycle orbits. The main point of the result is that the constant in the estimate is in terms of an “average injectivity radius” along the orbit, with respect to rescaled Riemannian metrics.

5.1. A priori bounds on ergodic integrals. Let $\lambda \in \mathbb{R}^*$ and let $\mathcal{T} \geq 1$. Let $(\phi_t^{\lambda, \mathcal{T}})_{t \in \mathbb{R}}$ denote the rescaled twisted horocycle flow on $M \times \mathbb{T}$, that is, the flow generated by the rescaled vector field $\mathcal{T}(U + \lambda K)$.

Let $\bar{x} \in M \times \mathbb{T}$, let $I \subset \mathbb{R}$ be an interval, and let $f \in C^\infty(M \times \mathbb{T})$. For $s > 1$, we will estimate in terms of the Sobolev norm $|f|_{0,s;\mathcal{T}}$ the ergodic integral

$$\left| \int_I f \circ \phi_t^{\lambda, \mathcal{T}}(\bar{x}) dt \right|.$$

We will follow the discussion of a Sobolev trace theorem in Section 3 of [8] that concerns nilmanifolds in particular. We will see that the method introduced there also gives the corresponding trace theorem in the $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{T}$ setting.

Let $\Delta_{\mathbb{R}^3}$ be the Euclidean Laplacian operator on \mathbb{R}^3 given by

$$\Delta_{\mathbb{R}^3} := - \left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial \theta^2} \right).$$

Given an open set $O \subset \mathbb{R}^3$ containing the origin, we consider the family \mathcal{R}_O of all 3-dimensional hyperrectangles $R \subset [-\frac{1}{2}, \frac{1}{2}]^3 \cap O$ that are centered at the origin. The *inner width* of the open set $O \subset \mathbb{R}^3$ is the positive number

$$w(O) := \sup\{\mathrm{Leb}(R) : R \in \mathcal{R}_O\},$$

where $\mathrm{Leb}(R)$ is the Lebesgue measure of R . The *width function* of an open set $\Omega \subset \mathbb{R} \times \mathbb{R}^3$ containing $\mathbb{R} \times \{0\}$ is the function defined for all $t \in \mathbb{R}$ by

$$w_\Omega(t) := w(\{y \in \mathbb{R}^3 : (t, y) \in \Omega\}).$$

We will now define average width, as in [7]. Let $\lambda > 0$ and $\mathcal{T} \geq 1$. Let $\bar{x} \in M \times \mathbb{T}$ and $T > 0$. Consider the family $\mathcal{O}_{\bar{x}, \lambda, \mathcal{T}, T}$ of open sets $\Omega \subset \mathbb{R} \times \mathbb{R}^3$ satisfying the following two conditions:

$$[0, T] \times \{0\} \subset \Omega \subset \mathbb{R} \times [-\frac{1}{2}, \frac{1}{2}]^3,$$

and the map $\alpha_{\bar{x}, \lambda, \mathcal{T}} : \mathbb{R}^3 \rightarrow M \times \mathbb{T}$ defined as

$$(78) \quad \begin{aligned} \alpha_{\bar{x}, \lambda, \mathcal{T}}(t, \theta, y, z) = & \bar{x} \exp(t\mathcal{T}(U + \lambda K)) \exp(y\mathcal{T}^{-1/3}X) \\ & \times \exp(z\mathcal{T}^{-2/3}V)) \exp(\theta K) \end{aligned}$$

is injective on the open set $\Omega \subset \mathbb{R}^3$.

The *average width* of the orbit segment of the twisted horocycle flow

$$\{\alpha_{\bar{x}, \lambda, \mathcal{T}, T}(t, 0, 0, 0) : 0 \leq t \leq T\}$$

is the positive number

$$w_{\mathcal{T}}(\bar{x}, \lambda, T) := \sup_{\Omega \in \mathcal{O}_{\bar{x}, \lambda, \mathcal{T}, T}} \left(\frac{1}{T} \int_0^T \frac{1}{w_{\Omega}(t)} dt \right)^{-1}.$$

We will estimate from below the average with $w_{\mathcal{T}}(\bar{x}, \lambda, T)$ of orbit segments of the twisted horocycle flow in the Section 6.1. In this section we derive a Sobolev trace theorem for orbit segments and a Sobolev embedding-type theorem with constants explicitly expressed in terms of the average width.

The following lemma is a special case of formula (32) of [8]. We prove it here for the convenience of the reader.

Lemma 5.1. *Let $I \subset \mathbb{R}$ be an interval, and let $\Omega \subset \mathbb{R} \times \mathbb{R}^3$ be a Borel set containing the segment $I \times \{0\} \subset \mathbb{R} \times \mathbb{R}^3$. For every $s > 1$, there is a constant $C_s > 0$ such that for all functions $F \in C^\infty(\Omega)$ and all $t \in I$, we have*

$$\left(\int_I |F(t, 0)| dt \right)^2 \leq C_s \left(\int_I \frac{1}{w_{\Omega}(t)} dt \right) \int_{\Omega} |(I - \Delta_{\mathbb{R}^3})^{s/2} F(t, y)|^2 dt dy.$$

Proof. Define $\Omega_t := \{y \in \mathbb{R}^3 : (t, y) \in \Omega\}$. By rescaling, Sobolev embedding gives

$$|F(t, 0)| \leq C_s w_{\Omega}(t)^{-1/2} \left(\int_{\Omega_t} |(I - \Delta_{\mathbb{R}^3})^{s/2} F(t, y)|^2 dy \right)^{1/2}.$$

Now we integrate over the interval I , and Hölder's inequality gives the result. \square

For the remainder of this section, we follow Lemma 3.7 and Theorem 3.9 of [8]. Contrary to [8], the vector fields $X_{\mathcal{T}}$, $V_{\mathcal{T}}$ and K that are transverse to the direction of integration do not all commute, but they still form an integrable distribution.

Theorem 5.2. *For all $s > 1$, there is a constant $C_s > 0$ such that the following holds. For $\lambda \in \mathbb{R}^*$, for all $\mathcal{T} \geq 1$, for all $\bar{x} \in M \times \mathbb{T}$, all $T > 0$ and for all $f \in C^\infty(M \times \mathbb{T})$, we have*

$$\left| \frac{1}{T} \int_0^T f \circ \phi_t^{\lambda, \mathcal{T}}(\bar{x}) dt \right| \leq C_s T^{-1/2} w_{\mathcal{T}}(\bar{x}, \lambda, T)^{-1/2} |f|_{0, s; \mathcal{T}}.$$

Proof. Let $\Omega \in \mathcal{O}_{\bar{x}, \lambda, \mathcal{T}, T}$. We have

$$\begin{aligned} \partial_{\theta}^2 f \circ \alpha_{\bar{x}, \lambda, \mathcal{T}}(t, y, z, \theta) &= K^2 f \circ \alpha_{\bar{x}, \lambda, \mathcal{T}}(t, y, z, \theta), \\ \partial_z^2 f \circ \alpha_{\bar{x}, \lambda, \mathcal{T}}(t, y, z, \theta) &= V_{\mathcal{T}}^2 f \circ \alpha_{\bar{x}, \lambda, \mathcal{T}}(t, y, z, \theta), \end{aligned}$$

and

$$\begin{aligned} \partial_y^2 f \circ \alpha_{\bar{x}, \lambda, \mathcal{T}}(t, y, z, \theta) &= (X_{\mathcal{T}} - z\mathcal{T}^{-1/3}V_{\mathcal{T}})^2 f \circ \alpha_{\bar{x}, \lambda, \mathcal{T}}(t, y, z, \theta) \\ &= [X_{\mathcal{T}}^2 + z^2\mathcal{T}^{-2/3}V_{\mathcal{T}}^2 - z\mathcal{T}^{-1/3}(X_{\mathcal{T}}V_{\mathcal{T}} + V_{\mathcal{T}}X_{\mathcal{T}})] f \circ \alpha_{\bar{x}, \lambda, \mathcal{T}}(t, y, z, \theta). \end{aligned}$$

Because X and V are essentially skew-adjoint,

$$0 \leq -(X_{\mathcal{T}} + V_{\mathcal{T}})^2 = -(X_{\mathcal{T}}^2 + V_{\mathcal{T}}^2) - (X_{\mathcal{T}}V_{\mathcal{T}} + V_{\mathcal{T}}X_{\mathcal{T}}).$$

Recall that $|z| \leq 1/2$, so

$$-V_{\mathcal{T}}^2 - (X_{\mathcal{T}} - z\mathcal{T}^{-1/3}V_{\mathcal{T}})^2 \leq 3(-X_{\mathcal{T}}^2 - V_{\mathcal{T}}^2),$$

Because the operators on the left and right are essentially self-adjoint, the spectral theorem gives

$$[I - K^2 - V_{\mathcal{T}}^2 - (X_{\mathcal{T}} - z\mathcal{T}^{-1/3}V_{\mathcal{T}})^2]^{s/2} \leq 3^{s/2}(I - K^2 - X_{\mathcal{T}}^2 - V_{\mathcal{T}}^2)^{s/2},$$

for any $s \geq 0$. Next observe that

$$\det(D\alpha_{\bar{x},\lambda,\mathcal{T}}(t,y,z,\theta)) = e^{-2y\mathcal{T}^{-1/3}}.$$

Then there is a constant $C_s > 0$ such that

$$(79) \quad \|(I - \Delta_{\mathbb{R}^3})^{s/2} f \circ \alpha_{\bar{x},\lambda,\mathcal{T}}\|_{L^2(\Omega)}^2 \leq C_s \|(I - K^2 - X_{\mathcal{T}}^2 - V_{\mathcal{T}}^2)^{s/2} f\|_{L^2(M)}^2.$$

By Lemma 5.1 and formula (79), we see that for any $s > 1$,

$$\begin{aligned} \left| \frac{1}{T} \int_0^T f \circ \phi_t^{\lambda,\mathcal{T}}(\bar{x}) dt \right|^2 &\leq \left(\frac{1}{T} \int_0^T |f \circ \alpha_{\bar{x},\lambda,\mathcal{T}}(t,0,0,0)| dt \right)^2 \\ &\leq C_s \frac{1}{T} \left(\frac{1}{T} \int_0^T \frac{1}{w_{\Omega}(t)} dt \right) \int_{\Omega} |(I - \Delta_{\mathbb{R}^3})^{s/2} f \circ \alpha_{\bar{x},\lambda,\mathcal{T}}(t,y,z,\theta)|^2 d\text{vol} \\ &\leq C_s T^{-1} w_{\mathcal{T}}(\bar{x},\lambda,T)^{-1} \|(I - K^2 - X_{\mathcal{T}}^2 - V_{\mathcal{T}}^2)^{s/2} f\|_{L^2(M)}^2. \end{aligned}$$

Because this holds for any set $\Omega \in \mathcal{O}_{\bar{x},\lambda,\mathcal{T},T}$, we may take the infimum over all sets $\Omega \in \mathcal{O}_{\bar{x},\lambda,\mathcal{T},T}$ and conclude the proof of Theorem 5.2. \square

5.2. Pointwise bounds for transfer functions. Following Lemma 3.7 and Theorem 3.9 of [8], we derive the following bound on transfer functions of the twisted horocycle flow.

Theorem 5.3. *Let $s > 1$. Then there is a constant $C_s > 0$ such that for all $\lambda \in \mathbb{R}^*$ and $\mathcal{T} \geq 1$, for all $f \in C^\infty(M \times \mathbb{T})$, if*

$$\mathcal{T}(U + \lambda K)g = f.$$

then for all $\bar{x} \in M \times \mathbb{T}$, $T > 0$ and for all $t \in [0, T]$,

$$|g \circ \phi_t^{\lambda,\mathcal{T}}(\bar{x})| \leq C_s T^{-1/2} w_{\mathcal{T}}(\bar{x},\lambda,T)^{-1/2} (T|f|_{0,s;\mathcal{T}} + |g|_{0,s;\mathcal{T}}).$$

Proof. Since $f \in H^\infty$, Theorems 3.4 and 3.18 imply that $g \in H^\infty$. Let $t \in [0, T]$. By the mean value theorem and by the fundamental theorem of calculus there exists $t_0 := t_0(\bar{x}, g) \in (0, T)$ such that

$$\begin{aligned} |g \circ \phi_t^{\lambda,\mathcal{T}}(\bar{x})| &= \left| \int_{t_0}^t \frac{d}{d\tau} g \circ \phi_\tau^{\lambda,\mathcal{T}}(\bar{x}) d\tau + \frac{1}{T} \int_0^T g \circ \phi_\tau^{\lambda,\mathcal{T}}(\bar{x}) d\tau \right| \\ &\leq \left| \int_{t_0}^t \mathcal{T}(U + \lambda K)g \circ \phi_\tau^{\lambda,\mathcal{T}}(\bar{x}) d\tau \right| + \left| \frac{1}{T} \int_0^T g \circ \phi_\tau^{\lambda,\mathcal{T}}(\bar{x}) d\tau \right| \\ &= \left| \int_{t_0}^t f \circ \phi_\tau^{\lambda,\mathcal{T}}(\bar{x}) d\tau \right| + \left| \frac{1}{T} \int_0^T g \circ \phi_\tau^{\lambda,\mathcal{T}}(\bar{x}) d\tau \right|. \end{aligned}$$

Now for $s > 1$, Theorem 5.2 implies Theorem 5.3. \square

6. TWISTED HOROCYCLE FLOWS: EFFECTIVE EQUIDISTRIBUTION

6.1. Average width function. In this section we estimate the average width for horocycle segments, which we define below. Let $x \in M$, $\mathcal{T} \geq 1$ and $T > 0$. Consider the family $\mathcal{O}_{x,\mathcal{T},T}$ of open sets $\Omega \subset \mathbb{R} \times \mathbb{R}^2$ satisfying the following two conditions:

$$[0, T] \times \{0\} \subset \Omega \subset \mathbb{R} \times \left[-\frac{1}{2}, \frac{1}{2}\right]^2,$$

and the map $\alpha_{x,\mathcal{T}} : \mathbb{R}^3 \rightarrow M \times \mathbb{T}$ defined as

$$(80) \quad \alpha_{x,\mathcal{T}}(t, y, z) = x \exp(t\mathcal{T}U) \exp(y\mathcal{T}^{-1/3}X) \exp(z\mathcal{T}^{-2/3}V)$$

is injective on the open set $\Omega \subset \mathbb{R}^2$.

The *average width* of the orbit segment of the horocycle flow

$$\{\alpha_{x,\mathcal{T},T}(t, 0, 0) : 0 \leq t \leq T\}$$

is the positive number

$$w_{\mathcal{T}}(x, T) := \sup_{\Omega \in \mathcal{O}_{x,\mathcal{T},T}} \left(\frac{1}{T} \int_0^T \frac{1}{w_{\Omega}(t)} dt \right)^{-1}.$$

We remark that by our definitions, since the twisted horocycle flow projects onto the horocycle flow under the projection of $M \times \mathbb{T}$ onto M and the vector field K tangent to the circle factor is not scaled, the average width for an orbit segment of the twisted horocycle flow is bounded below by the average for the projected orbit segment of the horocycle flow. In fact, the following holds:

Lemma 6.1. *For all $\bar{x} = (x, \theta)$, for all $\lambda \in \mathbb{R}$, for all $\mathcal{T} \geq 1$ and all $T > 0$, we have the inequality*

$$(81) \quad w_{\mathcal{T}}(\bar{x}, \lambda, T) \geq w_{\mathcal{T}}(x, T).$$

Proof. For any open set $\Omega \subset M$, let $\hat{\Omega} := \Omega \times [-1/2, 1/2]$. Since the vector field K commutes with the vector fields X, U, V and is tangent to the circle factor of the product $M \times \mathbb{T}$, it follows that for any $\bar{x} = (x, \theta) \in M \times \mathbb{T}$, for all $\mathcal{T} \geq 1$ and $T > 0$, the function $\alpha_{\bar{x},\lambda,\mathcal{T},T}$, defined in (78), is injective on $\hat{\Omega}$ if and only if the function $\alpha_{x,\mathcal{T},T}$, defined in (80), is injective on Ω . The statement then follows from the definitions. \square

It is therefore enough to estimate the average width $w_{\mathcal{T}}(x, T)$ of the orbits segments of the horocycle flow on M .

For any $x \in M$, we consider the map α_x defined on \mathbb{R}^3 by

$$\alpha_x(t, y, z) = x \exp(tU) \exp(yX) \exp(zV).$$

Definition 6.2. *For any $(x, T) \in M \times \mathbb{R}^+$, let $c_{\Gamma}(x, T)$ denote the smallest positive number $c \geq 1$ such that the map α_x is injective on the interval*

$$[-10T, 10T] \times [-1/2, 1/2] \times \frac{1}{c} \left[-\frac{1}{T}, \frac{1}{T}\right].$$

Remark 6.3. *By the action of the geodesic flow, it follows from the definition that for all $(x, T) \in M \times \mathbb{R}^+$ and for all $y \in \mathbb{R}$ we have*

$$c_\Gamma(x, T) = c_\Gamma(a_y(x), e^{-y}T).$$

The statement of the equidistribution theorems, Theorem 2.3 and Theorem 2.7, involve the function $C_\Gamma(x, T)$ defined for all $(x, T) \in M \times \mathbb{R}^+$ by

$$(82) \quad C_\Gamma(x, T) := \sup_{1 \leq t \leq T} c_\Gamma(x, t) = \sup_{-\log T \leq y \leq 0} c_\Gamma(a_y(x), T).$$

We begin by proving upper bounds on the function $c_\Gamma(x, T)$ under Diophantine conditions. For $A \in [0, 1)$ and $Q > 0$, recall that we consider subsets of “Diophantine points” given by

$$M_{A,Q} := \{x \in M : d_M(a_y(x)) \leq Ay + Q \text{ for all } y > 0\}.$$

In addition, by the logarithmic law of geodesics for almost all $x \in M$ and for all $\varepsilon > 0$ there exists a constant $Q_\varepsilon(x) > 0$ such that, for all $t \geq 1$,

$$d_M(a_y(x)) \leq \left(\frac{1}{2} + \varepsilon\right) \log y + Q_\varepsilon(x).$$

For all $A > 1/2$ and $Q, y_0 > 0$, we will therefore introduce the sets

$$\tilde{M}_{A,Q,y_0} := \{x \in M : d_M(a_y(x)) \leq A \log(y + y_0) + Q \text{ for all } y \geq 1 - y_0\}.$$

The proof of the following basic lemma is left for Appendix C.

Lemma 6.4. *For all $x \in M$, let $d_M(x) := \text{dist}(x, \Gamma)$. There exists a constant $C_\Gamma \in (0, 1)$ such that for all $x \in M$, the map $\alpha_x : \mathbb{R}^3 \rightarrow M$ defined by the formula*

$$\alpha_x(t, y, z) = x \exp(tU) \exp(yX) \exp(zV),$$

is injective on the interval

$$[-C_\Gamma e^{-d_M(x)}, C_\Gamma e^{-d_M(x)}] \times [-1/2, 1/2] \times [-C_\Gamma e^{-d_M(x)}, C_\Gamma e^{-d_M(x)}].$$

For all $x \in M$ and all $t > 0$, let

$$d_M(x, t) := \max_{0 \leq y \leq t} d_M(a_y(x)).$$

Lemma 6.5. *For all $x \in M$, for all $t > 0$ and $T \in [1, 10^{-1} C_\Gamma e^{t-d_M(x)})]$ we have*

$$c_\Gamma(x, T) \leq \left(\frac{10}{C_\Gamma}\right)^2 e^{2d_M(x,t)};$$

in particular for all $x \in M$ with bounded forward geodesic orbit, which is always the case whenever M is compact, and for all $T > 0$ we have

$$c_\Gamma(x, T) \leq \left(\frac{10}{C_\Gamma}\right)^2 \max_{x \in M} e^{2d_M(x)}.$$

For all $x \in M_{A,Q}$ and for all $T \geq 1$ we have the estimate

$$c_\Gamma(x, T) \leq (10C_\Gamma^{-1}e^Q)^{\frac{2}{1-A}} T^{\frac{2A}{1-A}}.$$

For every $A > 1/2$ there exists a constant $C_{\Gamma,A} > 0$ such that if $x \in \tilde{M}_{A,Q,y_0}$ then for all $T \geq 1$ we have

$$c_{\Gamma}(x, T) \leq C_{\Gamma,A} e^{2Q} (1 + Q + y_0 + \log T)^{2A}.$$

Proof. Let $C_{\Gamma} > 0$ be as in Lemma 6.4. Let x, t and T as in the statement of the Lemma and set

$$y_* = \min\{y > 0 \mid e^{-y} 10T \leq C_{\Gamma} e^{-d_M(a_y(x))}\},$$

We remark that, since

$$e^{-t} 10T \leq C_{\Gamma} e^{-d_M(a_t(x))},$$

we have $y_* \leq t$, as the set in the definition of y_* contains $t > 0$. In fact have

$$(83) \quad y_* \leq \log\left(\frac{10T}{C_{\Gamma}}\right) + d_M(x, t).$$

By definition of the positive real number y_* we have

$$e^{-y_*} 10T = C_{\Gamma} e^{-d_M(a_{y_*}(x))}.$$

By Lemma 6.4 the map $\alpha_{a_{y_*}(x)}$ is injective on

$$[-e^{-y_*} 10T, e^{-y_*} 10T] \times [-1/2, 1/2] \times [-e^{-y_*} 10T, e^{-y_*} 10T].$$

Recall that for any $t \in \mathbb{R}$, we have the commutation relations,

$$a_{-y_*} \circ h_t \circ a_{y_*} = h_{te^{y_*}};$$

$$a_{-y_*} \circ \bar{h}_t \circ a_{y_*} = \bar{h}_{te^{-y_*}}.$$

Then by right multiplication of a_{-y_*} on the image of $\alpha_{a_{y_*}(x)}$ restricted to the above set, we get that the map α_x is injective on the interval

$$[-10T, 10T] \times [-1/2, 1/2] \times \frac{T^2}{e^{2y_*}} \left[-\frac{1}{T}, \frac{1}{T}\right].$$

By the estimate in formula (83) it then follows that

$$c_{\Gamma}(x, T) \leq \frac{e^{2y_*}}{T^2} \leq \left(\frac{10}{C_{\Gamma}}\right)^2 e^{2d_M(x, t)}.$$

If x has bounded forward geodesic orbit we have in particular that, for all $T \geq 1$,

$$c_{\Gamma}(x, T) \leq \left(\frac{10}{C_{\Gamma}}\right)^2 \max_{x \in M} e^{2d_M(x)} < +\infty.$$

A straightforward estimate then shows that if $x \in M_{A,Q}$ then by the definitions we have

$$c_{\Gamma}(x, T) \leq \frac{e^{2y_*}}{T^2} \leq (10C_{\Gamma}^{-1} e^Q T)^{\frac{2}{1-A}} / T^2 = (10C_{\Gamma}^{-1} e^Q)^{\frac{2}{1-A}} T^{\frac{2A}{1-A}}.$$

Similarly if $x \in \tilde{M}_{A,Q,y_0}$, that is, if $d_M(a_y(x)) \leq A \log(y + y_0) + Q$ for all $y \geq 1 - y_0$, it follows by the definition that y_* is bounded above either by $A > 0$ or by the unique solution $Y \geq A$ of the identity

$$Y = A \log(Y + y_0) + Q + \log\left(\frac{10T}{C_{\Gamma}}\right).$$

By change of variable $Y + y_0$ is equal to the unique solution $Z \geq A + y_0$ of the equation

$$Z = A \log Z + Q + y_0 + \log\left(\frac{10T}{C_\Gamma}\right).$$

By a straightforward calculation there exists a constant $C_A > 0$ such that

$$Z \leq C_A + Q + y_0 + \log\left(\frac{10T}{C_\Gamma}\right) + A \log\left(Q + y_0 + \log\left(\frac{10T}{C_\Gamma}\right)\right).$$

We conclude that there exists a constant $C_{\Gamma,A} > 0$ such that

$$c_\Gamma(x, T) \leq \frac{e^{2y_*}}{T^2} \leq C_{\Gamma,A} e^{2Q} (1 + Q + y_0 + \log T)^{2A}.$$

The argument is concluded. \square

For any $x \in M$ and $\mathcal{T} \geq 1$, we consider the scaled map $\alpha_{x,\mathcal{T}}$ defined on \mathbb{R}^3 by

$$\alpha_{x,\mathcal{T}}(t, y, z) = x \exp(t\mathcal{T}U) \exp(y\mathcal{T}^{-1/3}X) \exp(z\mathcal{T}^{-2/3}V).$$

By the above definition and by change of variable we have the following statement.

Lemma 6.6. *For all $\mathcal{T} \geq 1$ and all $(x, T) \in M \times \mathbb{R}^+$, the map $\alpha_{x,\mathcal{T}}$ is injective on the interval*

$$[-10T, 10T] \times [-\mathcal{T}^{1/3}/2, \mathcal{T}^{1/3}/2] \times \frac{1}{c_\Gamma(x, \mathcal{T}T)} \left[-\frac{\mathcal{T}^{-1/3}}{T}, \frac{\mathcal{T}^{-1/3}}{T}\right].$$

Theorem 6.7. *There exists a constant $K_\Gamma > 0$ such that the following holds. For any $x \in M$, for any $T \geq 1$ and $\mathcal{T} \in [1, T]$, there is an open tubular neighborhood $\Omega_{\mathcal{T},T}(x)$ of $[0, T] \times \{(0, 0)\}$ in $[0, T] \times [-\frac{1}{2}, \frac{1}{2}]^2$ such that the map $\alpha_{x,\mathcal{T}} : \Omega_{\mathcal{T},T}(x) \rightarrow M$ is injective and the following estimate holds*

$$\frac{1}{T} \int_0^T \frac{1}{w_{\Omega_{\mathcal{T},T}}(t)} dt \leq K_\Gamma \cdot c_\Gamma^2(x, \mathcal{T}T) T (1 + \log(\mathcal{T}^{1/3}T)).$$

From the above theorem and from Lemma 6.1, we derive our main result on the average width of orbits segments of the twisted horocycle flow.

Corollary 6.8. *For any $\lambda \in \mathbb{R}^*$, for any $\bar{x} = (x, \theta) \in M \times \mathbb{T}$, for any $T \geq 1$ and for any $\mathcal{T} \in [1, T]$, we have the estimate*

$$w_{\mathcal{T}}(\bar{x}, \lambda, T)^{-1} \leq w_{\mathcal{T}}(x, T)^{-1} \leq K_\Gamma \cdot c_\Gamma^2(x, \mathcal{T}T) T (1 + \log(\mathcal{T}^{1/3}T)).$$

We prove below Theorem 6.7. A simple calculation shows

Lemma 6.9. *Let $t_0, s, z \in \mathbb{R}$ be such that $2|zs|\mathcal{T}^{1/3} < 1$. Then*

$$(84) \quad \begin{aligned} & \exp(t_0\mathcal{T}U) \exp(y\mathcal{T}^{-1/3}X) \exp(z\mathcal{T}^{-2/3}V) \exp(s\mathcal{T}U) \\ &= \exp(t_0(s)\mathcal{T}U) \exp(y(s)\mathcal{T}^{-1/3}X) \exp(z(s)\mathcal{T}^{-2/3}V), \end{aligned}$$

where

$$(85) \quad \begin{aligned} t_0(s) &= s e^{2y\mathcal{T}^{-1/3}} (1 + zs\mathcal{T}^{1/3})^{-1} + t_0 \\ y(s) &= y - \mathcal{T}^{1/3} \log(1 + \mathcal{T}^{1/3}zs) \\ z(s) &= z(1 + \mathcal{T}^{1/3}zs)^{-1}. \end{aligned}$$

We now introduce certain closest returns of horocycle orbits.

Definition 6.10. Let $\mathcal{T}, T \geq 1$. A pair $(t_0, t_1) \in [-10T, 10T]^2$ is called a (β, \mathcal{T}, T) -return for $x \in M$ if β is an integer in $[0, \log(\mathcal{T}^{1/3}T)]$ such that for some $|z| \in \frac{1}{c_{\mathcal{T}}(x, \mathcal{T}T)}(e^{-(\beta+1)}, e^{-\beta}]$ we have

$$x \exp(t_1 \mathcal{T}U) = x \exp(t_0 \mathcal{T}U) \exp(z \mathcal{T}^{-2/3}V).$$

The (β, \mathcal{T}, T) -return is called non-degenerate if $t_0 \neq t_1$ and degenerate if $t_0 = t_1$.

We denote by $n_{\mathcal{T}, T}^{\beta}(x)$ the number of non-degenerate (β, \mathcal{T}, T) -returns and by $n_{\mathcal{T}, T}^{\beta, \text{deg}}(x)$ the number of degenerate (β, \mathcal{T}, T) -returns for $x \in M$.

Let $\mathbb{H} := \{w \in \mathbb{C} \mid \text{Im}(w) > 0\}$ denote the upper half-plane. Let $\{C_i\}$ be the collection of disjoint cusps of the surface $S := \Gamma \backslash \mathbb{H}$ bounded by a cuspidal horocycles of length $\ell_{\Gamma} < 1$. By a cusp of M we mean the tangent unit bundle $\tilde{C}_i \subset M$ of a cusp C_i , that is, the pull-back to M of the cusp $C_i \subset S$. The manifold M can be decomposed as a disjoint union of a *thin part*, defined as the union of the finite collection of the disjoint cusps \tilde{C}_i , and of a compact *thick part*.

The following Lemma is proven in Appendix C.

Lemma 6.11. For any non-degenerate (β, \mathcal{T}, T) -return for $x \in M$ the return points $x \exp(t_1 \mathcal{T}U)$ and $x \exp(t_2 \mathcal{T}U)$ belong to the thick part of M . For any degenerate (β, \mathcal{T}, T) -return for $x \in M$ the point $x \exp(t_1 \mathcal{T}U) = x \exp(t_0 \mathcal{T}U)$ belongs to the cuspidal horocycle for the unstable horocycle $\{\tilde{h}_t\}$ generated by the vector field V on M .

Next lemma shows that non-degenerate (β, \mathcal{T}, T) -returns cannot be too close.

Lemma 6.12. There exists a constant $C_{\Gamma} > 1$ (depending only on the thick part of M) such that the following holds. Let $(t_0, t_1) \in [-10T, 10T]^2$ be a non-degenerate (β, \mathcal{T}, T) -return for $x \in M$. If $(t'_0, t'_1) \in [-10T, 10T]^2$ is another non-degenerate (β, \mathcal{T}, T) -return for x and

$$|t'_0 - t_0| < \frac{1}{2C_{\Gamma}} e^{\beta} \mathcal{T}^{-1/3}, \quad |t'_1 - t_1| < \frac{1}{2C_{\Gamma}} e^{\beta} \mathcal{T}^{-1/3},$$

then $t'_0 = t_0$ and $t'_1 = t_1$.

Proof. If (t_0, t_1) and (t'_0, t'_1) are (β, \mathcal{T}, T) -return pairs in $[-10T, 10T]^2$, by definition there exist z and z' with $|z|, |z'| \in \frac{1}{c_{\mathcal{T}}(x, \mathcal{T}T)}(e^{-(\beta+1)}, e^{-\beta}]$ such that

$$\begin{aligned} x \exp(t_1 \mathcal{T}U) &= x \exp(t_0 \mathcal{T}U) \exp(z \mathcal{T}^{-2/3}V), \\ x \exp(t'_1 \mathcal{T}U) &= x \exp(t'_0 \mathcal{T}U) \exp(z' \mathcal{T}^{-2/3}V). \end{aligned} \tag{86}$$

By geodesic scaling for geodesic time $\sigma := \beta + \frac{2}{3} \log \mathcal{T}$ of the identities in formula (86) we derive

$$\begin{aligned} a_{\sigma}(x) \exp(e^{-\sigma} t_1 \mathcal{T}U) &= a_{\sigma}(x) \exp(e^{-\sigma} t_0 \mathcal{T}U) \exp(e^{\sigma} z \mathcal{T}^{-2/3}V), \\ a_{\sigma}(x) \exp(e^{-\sigma} t'_1 \mathcal{T}U) &= a_{\sigma}(x) \exp(e^{-\sigma} t'_0 \mathcal{T}U) \exp(e^{\sigma} z' \mathcal{T}^{-2/3}V). \end{aligned}$$

The pairs $(e^{-\sigma}t_0\mathcal{T}, e^{-\sigma}t'_0\mathcal{T})$ and $(e^{-\sigma}t_1\mathcal{T}, e^{-\sigma}t'_1\mathcal{T})$ are non-degenerate $(0, 1, \mathcal{T})$ -returns for the point $a_\sigma(x)$ with

$$(87) \quad e^{-\sigma}|t'_0 - t_0|\mathcal{T} < \frac{1}{2C_\Gamma}, \quad e^{-\sigma}|t'_1 - t_1|\mathcal{T} < \frac{1}{2C_\Gamma}.$$

By Lemma 6.11 the points

$$\begin{aligned} a_\sigma(x) \exp(e^{-\sigma}t_0\mathcal{T}U), \quad a_\sigma(x) \exp(e^{-\sigma}t'_0\mathcal{T}U) \\ a_\sigma(x) \exp(e^{-\sigma}t_1\mathcal{T}U), \quad a_\sigma(x) \exp(e^{-\sigma}t'_1\mathcal{T}U) \end{aligned}$$

all belong to the thick part of M . Let $s = t'_1 - t_1$. Since $|s| \leq \frac{e^\beta}{2C_\Gamma}\mathcal{T}^{-1/3}$, we have

$$(88) \quad \mathcal{T}^{1/3}|zs| \leq \frac{e^\beta}{2C_\Gamma} \frac{e^{-\beta}}{c_\Gamma(x, \mathcal{T})} < \frac{1}{2C_\Gamma}.$$

Hence, by Lemma 6.9 we obtain

$$\begin{aligned} a_\sigma(x) \exp(e^{-\sigma}t'_1\mathcal{T}U) &= a_\sigma(x) \exp(e^{-\sigma}t_1\mathcal{T}U) \exp(e^{-\sigma}s\mathcal{T}U) \\ &= a_\sigma(x) \exp(e^{-\sigma}t_0\mathcal{T}U) \exp(e^\sigma z\mathcal{T}^{-2/3}V) \exp(e^{-\sigma}s\mathcal{T}U) \\ &= a_\sigma(x) \exp(e^{-\sigma}t_0(s)\mathcal{T}U) \exp(y(s)\mathcal{T}^{-1/3}X) \exp(e^\sigma z(s)\mathcal{T}^{-2/3}V), \end{aligned}$$

where $t_0(s)$, $y(s)$ and $z(s)$ are given by formulas (85) with $y = 0$, that is,

$$(89) \quad \begin{aligned} t_0(s) &= s(1 + \mathcal{T}^{1/3}zs)^{-1} + t_0 \\ y(s) &= -\mathcal{T}^{1/3} \log(1 + \mathcal{T}^{1/3}zs) \\ z(s) &= z(1 + \mathcal{T}^{1/3}zs)^{-1}. \end{aligned}$$

It follows that

$$(90) \quad \begin{aligned} a_\sigma(x) \exp(e^{-\sigma}t'_0\mathcal{T}U) \exp(e^\sigma z'\mathcal{T}^{-2/3}V) \\ = a_\sigma(x) \exp(e^{-\sigma}t_0(s)\mathcal{T}U) \exp(y(s)\mathcal{T}^{-1/3}X) \exp(e^\sigma z(s)\mathcal{T}^{-2/3}V). \end{aligned}$$

By the estimate (88) the above expression for $t_0(s)$, $y(s)$ and $z(s)$,

$$(91) \quad \begin{aligned} t_0(s) &\in [t_0 - 2s, t_0 + 2s], \\ y(s) &\in [-\mathcal{T}^{1/3}/2, \mathcal{T}^{1/3}/2], \\ |z(s)| &\in [z/2, 2z]. \end{aligned}$$

Let now $x_\sigma \in M$ denote the intermediate point

$$x_\sigma := a_\sigma(x) \exp(e^{-\sigma} \frac{t'_0 + t_0(s)}{2} \mathcal{T}U).$$

By the above bounds, since the points

$$a_\sigma(x) \exp(e^{-\sigma}t_0\mathcal{T}U) \quad \text{and} \quad a_\sigma(x) \exp(e^{-\sigma}t'_0\mathcal{T}U)$$

belong to the thick part of M , it follows that x_σ belongs to the compact set of points at distance at most $1/C_\Gamma$ from the thick part.

The identity in formula (90) can be rewritten as

$$\begin{aligned}
 (92) \quad & x_\sigma \exp(e^{-\sigma} \frac{t'_0 - t_0(s)}{2} \mathcal{T}U) \exp(e^\sigma z' \mathcal{T}^{-2/3} V) \\
 & = x_\sigma \exp(e^{-\sigma} \frac{t_0(s) - t_0}{2} \mathcal{T}U) \exp(y(s) \mathcal{T}^{-1/3} X) \exp(e^\sigma z(s) \mathcal{T}^{-2/3} V).
 \end{aligned}$$

Since x_σ is at distance at most $1/C_\Gamma$ from the thick part and by the bounds in formulas (87) and (91), it follows that there exists a constant $C_\Gamma > 1$ for which the identity (92) implies that

$$t_0(s) - t'_0 = y(s) = z(s) - z' = 0.$$

Formula (89) shows that the $y(s) = 0$ implies that $sz = 0$ and consequently $s = t_1 - t'_1 = 0$ since $z \neq 0$ and $t_0(s) = t_0$ and $z(s) = z$. In turn this implies $t'_0 = t_0$ and $z = z'$. The argument is now concluded. \square

As a consequence, we derive the following bound for the number $n_{\mathcal{T},T}^\beta(x)$ of non-degenerate (β, \mathcal{T}, T) -returns.

Proposition 6.13. *We have*

$$n_{\mathcal{T},T}^\beta(x) \leq 4 \cdot 10^2 C_\Gamma^2 e^{-2\beta} \mathcal{T}^{2/3} T^2.$$

Proof. By Lemma 6.12, the maximum number of disjoint squares of side length $\frac{1}{2C_\Gamma} e^\beta \mathcal{T}^{-1/3}$ that fit inside the square $[-10T, 10T]^2$ is bounded by

$$\frac{(10T)^2}{(e^\beta \mathcal{T}^{-1/3} / 2C_\Gamma)^2} \leq 4 \cdot 10^2 C_\Gamma^2 e^{-2\beta} \mathcal{T}^{2/3} T^2.$$

\square

The number $n_{\mathcal{T},T}^{\beta,deg}(x)$ of degenerate (β, \mathcal{T}, T) -returns is estimated as follows.

Proposition 6.14. *There exists a constant $C'_\Gamma > 0$ such that*

$$n_{\mathcal{T},T}^{\beta,deg}(x) \leq C'_\Gamma (1 + e^{-\beta} \mathcal{T}^{1/3} T).$$

Proof. There exists a constant $c_\Gamma > 0$ such that the following holds. By definition and by Lemma 6.11, for any degenerate (β, \mathcal{T}, T) -return $t_0 = t_1$ for $x \in M$, the point $x \exp(t_0 \mathcal{T}U)$ must be a point of the unstable cuspidal horocycle at distance from the thick part of M larger than

$$c_\Gamma (\beta + \frac{2}{3} \log \mathcal{T}).$$

It is therefore enough to bound the number of points of intersections of a stable horocycle arc of length $T > 0$ with the unstable cuspidal horocycles at a distance larger than $d > 0$ from the thick part. We claim is that there are at most $1 + eT/e^d$. The statement will then follow immediately from the claim since we are counting the degenerate close returns of a horocycle arc of rescaled length $T > 0$, hence of hyperbolic length $\mathcal{T}T > 0$.

By applying the geodesic flow for a time $t = d - 1$ we get a horocycle arc of hyperbolic length eT/e^d . The points of intersection of the stable horocycle of

hyperbolic length $T > 0$ with the unstable cuspidal horocycle which are at distance larger than d from the thick part are sent by the geodesic time map to points of intersection of the shortened stable horocycle of length eT/e^d with the unstable cuspidal horocycle which are at distance larger than 1 from the thick part. Since between any two such intersections the stable horocycle has to enter the thick part, their total number is at most $1 + eT/e^d$. The argument is therefore completed. \square

We now construct tubular neighborhoods $\Omega_{\mathcal{T},T}(x)$ of $[0, T] \times \{(0, 0)\}$ in $[0, T] \times [-1/2, 1/2]^2$ with the properties claimed in Theorem 6.7.

Suppose that $t_0 \in [-10T, 10T]$ belongs to a (β, \mathcal{T}, T) -return pair. Let

$$\Omega_{\mathcal{T},T}^{\beta,t_0} \subset \left([-10T, 10T] \cap [t_0 - e^\beta \mathcal{T}^{-2/3}, t_0 + e^\beta \mathcal{T}^{-2/3}] \right) \times \left[-\frac{1}{2}, \frac{1}{2} \right]^2$$

be a partial tubular neighborhood of $[-10T, 10T] \times \{(0, 0)\}$ defined locally about t_0 whose cross-section at a point $t \in [-10T, 10T] \cap [t_0 - e^\beta \mathcal{T}^{-2/3}, t_0 + e^\beta \mathcal{T}^{-2/3}]$ is a square centered at $(0, 0)$ contained in $[-\frac{1}{2}, \frac{1}{2}]^2$ with side-length equal to

$$(93) \quad \frac{1}{100} \frac{e^{-\beta}}{c_\Gamma(x, \mathcal{T}T)} \max\{\mathcal{T}^{2/3}|t - t_0|, 1\}.$$

Extend then the tube $\Omega_{\mathcal{T},T}^{\beta,t_0}$ to a full tubular neighborhood $\Omega_{\mathcal{T},T}^{\beta,t_0,E}$ of $[-10T, 10T] \times \{(0, 0)\}$ whose cross-sections are squares centered at $(0, 0)$ with side-length equal to $\frac{1}{100} \frac{1}{c_\Gamma(x, \mathcal{T}T)}$, for all $t \in [-10T, 10T] \setminus [t_0 - e^\beta \mathcal{T}^{-2/3}, t_0 + e^\beta \mathcal{T}^{-2/3}]$. Let then

$$\mathcal{A}_{\mathcal{T},T}(x) := \{(\beta, t_0) \in \mathbb{N} \times [-10T, 10T] : t_0 \text{ belongs to a } (\beta, \mathcal{T}, T)\text{-return pair}\},$$

and define $\Omega_{\mathcal{T},T}(x) \subset [0, T] \times [-1/2, 1/2]^2$ by

$$\Omega_{\mathcal{T},T}(x) := \bigcap_{(\beta, t_0) \in \mathcal{A}_{\mathcal{T},T}(x)} \Omega_{\mathcal{T},T}^{\beta,t_0,E} \cap [0, T] \times [-1/2, 1/2]^2.$$

Lemma 6.15. *The map $\alpha_{x,\mathcal{T}} : \Omega_{\mathcal{T},T}(x) \rightarrow M$ is injective.*

Proof. Suppose that $(t_0, y_0, z_0), (t_1, y_1, z_1) \in \Omega_{\mathcal{T},T}(x)$ are such that

$$\alpha_{x,\mathcal{T}}(t_0, y_0, z_0) = \alpha_{x,\mathcal{T}}(t_1, y_1, z_1).$$

From this condition we will derive that $(t_0, y_0, z_0) = (t_1, y_1, z_1)$.

Let $y_2 := y_0 - y_1$ and $z_2 := (z_0 - z_1)e^{y_1 \mathcal{T}^{-1/3}}$. A calculation shows

$$(94) \quad x \exp(t_1 \mathcal{T}U) = x \exp(t_0 \mathcal{T}U) \exp(y_2 \mathcal{T}^{-1/3}X) \exp(z_2 \mathcal{T}^{-2/3}V).$$

From the definition of the tube $\Omega_{\mathcal{T},T}(x)$, we have, for $i = 0, 1$,

$$|y_i|, \quad |z_i| \leq \frac{1}{100} \frac{1}{c_\Gamma(x, \mathcal{T}T)}.$$

which by the above formulas for y_2 and z_2 implies that

$$(95) \quad |y_2| \leq \frac{1}{50} \frac{1}{c_\Gamma(x, \mathcal{T}T)}, \quad |z_2| \leq \frac{1}{25} \frac{1}{c_\Gamma(x, \mathcal{T}T)}.$$

We therefore assume that $(t_0, y_0, z_0) \neq (t_1, y_1, z_1)$ and derive a contradiction.

Since by assumption $t_0, t_1 \in [0, T]$, from Lemma 6.6 and formula (94) it follows that $(t_0, y_0, z_0) = (t_1, y_1, z_1)$ whenever $|z_2| < \frac{1}{c_\Gamma(x, \mathcal{T}T)} \frac{\mathcal{T}^{-1/3}}{T}$. Since by our assumption $(t_0, y_0, z_0) \neq (t_1, y_1, z_1)$, it follows that

$$|z_2| \in \frac{1}{c_\Gamma(x, \mathcal{T}T)} \left[\frac{\mathcal{T}^{-1/3}}{T}, 1 \right].$$

Then there is a number $\beta \in \mathbb{Z} \cap [0, \log(\mathcal{T}^{1/3}T)]$ such that

$$(96) \quad |z_2| \in \frac{1}{c_\Gamma(x, \mathcal{T}T)} (e^{-(\beta+1)}, e^{-\beta}].$$

By formulas (84) and (94), on the interval $I_{\mathcal{T}, \beta} := \{s : \mathcal{T}^{1/3}|z_2 s| < 7/8\}$ we can write

$$x \exp((t_1 + s)\mathcal{T}U) = x \exp(t_0(s)\mathcal{T}U) \exp(y_2(s)\mathcal{T}^{-1/3}X) \exp(z_2(s)\mathcal{T}^{-2/3}V),$$

where $t_0(s)$, $y_2(s)$ and $z_2(s)$ are given by formula (85), that is,

$$(97) \quad \begin{aligned} t_0(s) &= s e^{2y_2 \mathcal{T}^{-1/3}} (1 + \mathcal{T}^{1/3} z_2 s)^{-1} + t_0 \\ y_2(s) &= y_2 - \mathcal{T}^{1/3} \log(1 + \mathcal{T}^{1/3} z_2 s) \\ z_2(s) &= z_2 (1 + \mathcal{T}^{1/3} z_2 s)^{-1}. \end{aligned}$$

A calculation based on Taylor formula or the intermediate value theorem shows that there is a smooth function h_{z_2} on $I_{\mathcal{T}, \beta}$ such that

$$(98) \quad y_2(s) = y_2 - \mathcal{T}^{2/3} z_2 s + h_{z_2}(s).$$

and

$$(99) \quad |h_{z_2}(s)| \leq 4\mathcal{T}|z_2 s|^2.$$

So if $y_2 < 0$, then for s satisfying $|s| = \frac{1}{20} \mathcal{T}^{-2/3} e^{\beta+1}$ with $z_2 s < 0$, since $\mathcal{T} \geq 1$, we have

$$y_2 - \mathcal{T}^{2/3} z_2 s + h_{z_2}(s) \geq \frac{1}{c_\Gamma(x, \mathcal{T}T)} \left(-\frac{1}{50} + \frac{1}{20} - \frac{1}{50} \right) > 0.$$

Because $y_2(0) = y_2 < 0$ and h_{z_2} is continuous, it follows that there is

$$(100) \quad s^* \in [-\mathcal{T}^{-2/3} e^{\beta+1}/20, \mathcal{T}^{-2/3} e^{\beta+1}/20] \subseteq [-10T, 10T]$$

such that

$$y_2(s^*) = 0.$$

A similar argument holds when $y_2 > 0$, with $|s| = \frac{1}{20} \mathcal{T}^{-2/3} e^{\beta+1}$ and $z_2 s > 0$.

The above formula for $z_2(s)$ gives

$$(101) \quad \frac{1}{c_\Gamma(x, \mathcal{T}T)} e^{-(\beta+2)} < |z_2(s^*)| \leq \frac{1}{c_\Gamma(x, \mathcal{T}T)} e^{-\beta+1}.$$

In other words, there is some $\delta \in \{\beta - 1, \beta, \beta + 1\}$ such that $(t_0(s^*), t_1 + s^*)$ is a δ -close pair. Now we will use the definitions of the open sets $\Omega_{\mathcal{T}, T}^{\delta, t_0(s^*)}$ and $\Omega_{\mathcal{T}, T}^{\delta, t_1 + s^*}$ to derive the contradiction that

$$(102) \quad (t_0, y_0, z_0) \notin \Omega_{\mathcal{T}, T}^{\delta, t_0(s^*)} \quad \text{or} \quad (t_1, y_1, z_1) \notin \Omega_{\mathcal{T}, T}^{\delta, t_1 + s^*}.$$

From formula (93), for all $s \in \mathbb{R}$ such that $|t_0(s) - t_0(s^*)| \leq e^\delta \mathcal{T}^{-2/3}$, let

$$E_{t_0(s^*)}^\delta(s) := \frac{1}{100} \frac{e^{-\delta}}{c_\Gamma(x, \mathcal{T}T)} \max\{\mathcal{T}^{2/3}|t_0(s) - t_0(s^*)|, 1\}$$

be the edge length of the cross-section at $(t_0(s), 0, 0)$ of the tube $\Omega_{\mathcal{T}, T}^{\delta, t_0(s^*)}$, and for all $s \in \mathbb{R}$ such that $|s - s^*| \leq e^\delta \mathcal{T}^{-2/3}$, let

$$E_{t_1+s^*}^\delta(s) := \frac{1}{100} \frac{e^{-\delta}}{c_\Gamma(x, \mathcal{T}T)} \max\{\mathcal{T}^{2/3}|s - s^*|, 1\}$$

be the edge length of the cross-section at $(t_1 + s, 0, 0)$ of the tube $\Omega_{\mathcal{T}, T}^{\delta, t_1+s^*}$.

By formulas (97) and (100) for all $s \in [0, s^*]$ and for $\delta \in \{\beta - 1, \beta, \beta + 1\}$ we have

$$(103) \quad |t_0 - t_0(s^*)| \leq 2|s - s^*| \leq 2|s^*| \leq e^\delta \mathcal{T}^{-2/3}.$$

In particular, the edge lengths at the points (t_0, y_0, z_0) and (t_1, y_1, z_1) are respectively $E_{t_0(s^*)}^\delta(0)$ and $E_{t_1+s^*}^\delta(0)$.

By the assumption that $(t_0, y_0, z_0), (t_1, y_1, z_1) \in \Omega_{\mathcal{T}, T}(x)$ and by formula (94) we deduce the inequality

$$(104) \quad \max\{|z_2|, |y_2|\} \leq E_{t_0(s^*)}^\delta(0) + E_{t_1+s^*}^\delta(0).$$

By the above expression and by formula (103) we derive the bound

$$(105) \quad \max\{|z_2|, |y_2|\} \leq \frac{1}{50} \frac{e^{-\delta}}{c_\Gamma(x, \mathcal{T}T)} \max\{2\mathcal{T}^{2/3}|s^*|, 1\}.$$

However, on the one hand if $|s^*| \leq \mathcal{T}^{-2/3}/2$, we derive the inequality

$$\frac{1}{c_\Gamma(x, \mathcal{T}T)} e^{-(\beta+1)} < |z_2| \leq \frac{1}{50} \frac{e^{-\delta}}{c_\Gamma(x, \mathcal{T}T)},$$

which cannot hold as $\delta \leq \beta + 1$, on the other hand if $|s^*| \geq \mathcal{T}^{-2/3}/2$ by formula (98) since $y_2(s^*) = 0$ we derive the inequality

$$|y_2| = |\mathcal{T}^{2/3} z_2 s^* + h_{z_2}(s^*)| < \frac{1}{25} \frac{e^{-\delta}}{c_\Gamma(x, \mathcal{T}T)} \mathcal{T}^{2/3} |s^*|,$$

which cannot hold as, by formulas (96) and (99), we have

$$|\mathcal{T}^{2/3} z_2 s^* + h_{z_2}(s^*)| \geq \frac{1}{2} |z_2| \mathcal{T}^{2/3} |s^*| \geq \frac{1}{2} \frac{e^{-(\beta+1)}}{c_\Gamma(x, \mathcal{T}T)} \mathcal{T}^{2/3} |s^*|.$$

Thus if $\alpha_{x, \mathcal{T}}(t_0, y_0, z_0) = \alpha_{x, \mathcal{T}}(t_1, y_1, z_1)$ for $(t_0, y_0, z_0), (t_1, y_1, z_1) \in \Omega_{\mathcal{T}, T}(x)$, under the assumption that $(t_0, y_0, z_0) \neq (t_1, y_1, z_1)$ we have reached a contradiction in all cases. It follows that $(t_0, y_0, z_0) = (t_1, y_1, z_1)$. Thus the map $\alpha_{x, \mathcal{T}}$ is injective on $\Omega_{\mathcal{T}, T}(x)$. This concludes the proof of Lemma 6.15. \square

Now we estimate the contribution to the average width function from each "pinched" regions $\Omega_{\mathcal{T}, T}^{\beta, t_0}$ of the tube $\Omega_{\mathcal{T}, T}(x)$. Let $w_{\Omega_{\mathcal{T}, T}^{\beta, t_0}}$ be the width function for $\Omega_{\mathcal{T}, T}^{\beta, t_0}$.

Lemma 6.16. *We have*

$$\int_{t_0 - e^\beta \mathcal{T}^{-2/3}}^{t_0 + e^\beta \mathcal{T}^{-2/3}} \frac{1}{w_{\Omega_{\mathcal{T},T}^{\beta,t_0}(t)}} dt \leq 4 \cdot 10^4 c_\Gamma^2(x, \mathcal{T}T) \mathcal{T}^{-2/3} e^{2\beta}.$$

Proof. A computation shows that, for all $\beta \geq 0$, $\mathcal{T} \geq 1$ we have the estimate

$$I_{\beta, \mathcal{T}} := \int_0^{\mathcal{T}^{-2/3}} \frac{1}{e^{-2\beta}} dt + \int_{\mathcal{T}^{-2/3}}^{+\infty} \frac{1}{e^{-2\beta} (\mathcal{T}^{2/3} t)^2} dt \leq 2 \mathcal{T}^{-2/3} e^{2\beta}.$$

By formula (93), we observe that

$$\begin{aligned} \int_{t_0 - e^\beta \mathcal{T}^{-2/3}}^{t_0 + e^\beta \mathcal{T}^{-2/3}} \frac{1}{w_{\Omega_{\mathcal{T},T}^{\beta,t_0}(t)}} dt &\leq 2 \cdot 10^4 c_\Gamma^2(x, \mathcal{T}T) I_{\beta, \mathcal{T}} \\ &\leq 4 \cdot 10^4 c_\Gamma^2(x, \mathcal{T}T) \mathcal{T}^{-2/3} e^{2\beta}. \end{aligned}$$

□

Proof of Theorem 6.7. By Lemma 6.16, we get

$$\begin{aligned} \frac{1}{T} \int_0^T \frac{1}{w_{\Omega_{\mathcal{T},T}(t)}} dt &\leq c_\Gamma^2(x, \mathcal{T}T) \\ &+ \frac{1}{T} \sum_{\beta=0}^{[\log(\mathcal{T}^{1/3}T)]+1} 4 \cdot 10^4 n_{\mathcal{T},T}^\beta(x) \mathcal{T}^{-2/3} c_\Gamma^2(x, \mathcal{T}T) e^{2\beta} \\ &+ \frac{1}{T} \sum_{\beta=0}^{[\log(\mathcal{T}^{1/3}T)]+1} 4 \cdot 10^4 n_{\mathcal{T},T}^{\beta,deg}(x) \mathcal{T}^{-2/3} c_\Gamma^2(x, \mathcal{T}T) e^{2\beta}. \end{aligned}$$

Now by Proposition 6.13 we have

$$\begin{aligned} \sum_{\beta=0}^{[\log(\mathcal{T}^{1/3}T)]+1} n_{\mathcal{T},T}^\beta(x) \mathcal{T}^{-2/3} e^{2\beta} &\leq 4 \cdot 10^2 C_\Gamma^2 \sum_{\beta=0}^{[\log(\mathcal{T}^{1/3}T)]+1} (e^{-2\beta} \mathcal{T}^{2/3} T^2) \mathcal{T}^{-2/3} e^{2\beta} \\ &\leq 4 \cdot 10^2 C_\Gamma^2 T^2 (1 + \log(\mathcal{T}^{1/3}T)). \end{aligned}$$

By Proposition 6.14 we have

$$\begin{aligned} \sum_{\beta=0}^{[\log(\mathcal{T}^{1/3}T)]+1} n_{\mathcal{T},T}^{\beta,deg}(x) \mathcal{T}^{-2/3} e^{2\beta} &\leq 4 \cdot 10^2 C_\Gamma^2 C'_\Gamma \sum_{\beta=0}^{[\log(\mathcal{T}^{1/3}T)]+1} (1 + e^{-\beta} \mathcal{T}^{1/3} T) \mathcal{T}^{-2/3} e^{2\beta} \\ &\leq 8 \cdot 10^2 e^2 C_\Gamma^2 C'_\Gamma T^2. \end{aligned}$$

The argument is therefore complete. □

6.2. The rescaling argument. In this section we prove Theorem 2.3. Let us recall that the twisted horocycle flow on $M \times \mathbb{T}$ projects to the horocycle flow on M . Because the equidistribution of the horocycle flow on M is well understood (see [2], [3], [6], [21]), we restrict our considerations to functions with zero average along the circle action on $M \times \mathbb{T}$. Such functions are in the orthogonal complement of functions constant along the circle action with respect to any of the Sobolev norms considered in this paper.

Let $r \geq 0$, $s > 1/2$, and let $T \geq 1$. Let us denote the ergodic integral (13) by

$$\gamma_{\bar{x}, \lambda}^T := \frac{1}{T} \int_0^T (\phi_t^\lambda(\bar{x}))^* dt.$$

We will estimate $\gamma_{\bar{x}, \lambda}^T$ by iteratively rescaling the (intermediate) Sobolev norms as in [7] and [8]. Let $h \in [1, 2]$ be a real number and $l \in \mathbb{N}$ be such that $T = e^{lh}$. For all integers $j \in [0, l]$, let

$$\mathcal{T}_j := e^{(l-j)h}, \quad T_j := T/\mathcal{T}_j = e^{jh}.$$

Then as j decreases from l to 0, the scaling parameter \mathcal{T}_j becomes larger, while the scaled length T_j of the arc $\{\phi_t^\lambda\}_{t=0}^{T_j}$ becomes shorter.

Let $(\phi_t^{\lambda, \mathcal{T}_j})$ again denote the flow of the scaled vector field $\mathcal{T}_j(U + \lambda K)$. Observe that by change of variable we have

$$\gamma_{\bar{x}, \lambda}^T = \frac{1}{T_j} \int_0^{T_j} (\phi_t^{\lambda, \mathcal{T}_j}(\bar{x}))^* dt.$$

Moreover, notice that if $\mathcal{D} \in \mathcal{I}_\lambda^s(\Gamma)$, then for all $\mathcal{T} \geq 1$,

$$\mathcal{T}(U + \lambda K)\mathcal{D} = 0.$$

Hence, the space $\mathcal{I}_\lambda^s(\Gamma)$ is independent of the scaling parameter.

By orthogonality it is enough to estimate $\gamma_{\bar{x}, \lambda}^T$ in each irreducible, unitary representation of $\widehat{W}^{-r, -s}(M \times \mathbb{T})$. For any $\mu \in \text{spec}(\square)$ and for any $m \in \mathbb{Z} \setminus \{0\}$, let $H := H_{m, \mu}$ be an irreducible, unitary representation of $\text{SL}(2, \mathbb{R}) \times \mathbb{T}$. The space $\widehat{H}_{\mathcal{T}_j}^{-r, -s}$ has an orthogonal decomposition

$$\widehat{H}_{\mathcal{T}_j}^{-r, -s} = (\mathcal{I}_\lambda^s(\Gamma) \cap \widehat{H}_{\mathcal{T}_j}^{-r, -s}) \oplus^\perp (\mathcal{I}_\lambda^s(\Gamma)^\perp \cap \widehat{H}_{\mathcal{T}_j}^{-r, -s}).$$

Then $\gamma_{\bar{x}, \lambda}^T$ has a corresponding orthogonal decomposition in $\widehat{H}_{\mathcal{T}_j}^{-r, -s}$ written as

$$(106) \quad \gamma_{\bar{x}, \lambda}^T|_{\widehat{H}^{-r, -s}} = \mathcal{D}^j \oplus^\perp \mathcal{R}^j$$

with

$$\begin{aligned} \mathcal{D}^j &:= \mathcal{D}_{x, \lambda, T, m, \mu, j}^{r, s} \in \mathcal{I}_\lambda^s(\Gamma) \cap \widehat{H}^{-r, -s}, \\ \mathcal{R}^j &:= \mathcal{R}_{x, \lambda, T, m, \mu, j}^{r, s} \in \mathcal{I}_\lambda^s(\Gamma)^\perp \cap \widehat{H}^{-r, -s}, \end{aligned}$$

We begin by estimating the scaled foliated Sobolev norms of the remainder distribution $\mathcal{R}^j \in \widehat{H}^{-r, -s}$.

Lemma 6.17. *Let $s > 2$ and let $r \geq 3(s-1)$. There is a constant $C_s'' := C_s''(\Gamma) > 0$ such that the following holds. For all $\lambda \in \mathbb{R}^*$, $m \in \mathbb{Z} \setminus \{0\}$, for all $\bar{x} = (x, \theta) \in M \times \mathbb{T}$ and for all $T \geq 1$, for all $j \in [1, l] \cap \mathbb{Z}$, the distribution $\mathcal{R}^j \in \hat{H}^{-r, -s}$ satisfies the estimate*

$$|\mathcal{R}^j|_{-r, -s; \mathcal{T}_j} \leq \frac{C_s''}{T_j} [c_\Gamma(x, \mathcal{T}_j) + c_\Gamma(h_T(x), \mathcal{T}_j)] \times (1 + \log^{1/2} \mathcal{T}_j) \frac{1 + |\lambda m|^{-(s-1)}}{|\lambda m|}.$$

Proof. Let $s > 2$, let $r \geq 3(s-1)$ and let $f \in H^\infty$. Let f^* be the orthogonal projection in $\hat{H}_{\mathcal{T}_j}^{-s}$ of f into $\text{Ann}(\mathcal{J}_\lambda^s(\Gamma))$. Because $f^* \in \text{Ann}_\lambda(\Gamma)$, by Theorem 3.4 and Theorem 3.18 there exists a solution $g^* \in \hat{H}^\infty$ of the equation

$$\mathcal{T}_j(U + \lambda K)g^* = f^*,$$

such that the following bounds holds:

$$(107) \quad |g^*|_{0, s-1; \mathcal{T}_j} \leq \frac{C_s}{\mathcal{T}_j^{1/3}} \frac{1 + |\lambda m|^{-(s-1)}}{|\lambda m|} |f|_{r, s; \mathcal{T}_j}.$$

By orthogonality, again we have

$$\mathcal{R}^j(f) = \gamma_{\bar{x}, \lambda}^T(f^*).$$

Then the fundamental theorem of calculus gives

$$\begin{aligned} |\mathcal{R}^j(f)| &= |\gamma_{\bar{x}, \lambda}^T(f^*)| \\ &= \left| \frac{1}{T_j} \int_0^{T_j} f^* \circ \phi_t^{\lambda, \mathcal{T}_j}(x) dt \right| \\ &= \left| \frac{1}{T_j} \int_0^{T_j} \mathcal{T}_j(U + K)g^* \circ \phi_t^{\lambda, \mathcal{T}_j}(x) dt \right| \\ (108) \quad &= \left| \frac{1}{T_j} \left(g^* \circ \phi_{T_j}^{\lambda, \mathcal{T}_j}(x) - g^*(x) \right) \right| \end{aligned}$$

By Theorem 5.3, for all $s > 2$ there exists a constant $C'_s > 0$ such that

$$(108) \leq \frac{C'_s}{T_j} [w_{\mathcal{T}_j}(x, \lambda, 1)^{-1/2} + w_{\mathcal{T}_j}(\phi_{T_j}^{\lambda, \mathcal{T}_j}(x), \lambda, 1)^{-1/2}] \times (|f^*|_{0, s-1; \mathcal{T}_j} + |g^*|_{0, s-1; \mathcal{T}_j}),$$

any by Corollary 6.8 there exists a constant $K_\Gamma > 0$ such that

$$\begin{aligned} &w_{\mathcal{T}_j}(x, \lambda, 1)^{-1/2} + w_{\mathcal{T}_j}(\phi_{T_j}^{\lambda, \mathcal{T}_j}(x), \lambda, 1)^{-1/2} \\ &\leq K_\Gamma [c_\Gamma(x, \mathcal{T}_j) + c_\Gamma(h_T(x), \mathcal{T}_j)] (1 + \log^{1/2} \mathcal{T}_j). \end{aligned}$$

Lemma 6.17 then follows from the above estimates. \square

Next, we estimate the scaled foliated norms of invariant distributions.

Lemma 6.18. *For every $s > 2$ and $r \geq 2s$, there is a constant $C_{r,s} > 0$ such that*

$$|\mathcal{D}^l|_{-r,-s} \leq C_{r,s} T^{-1/6} (1 + |\lambda m|^{-3s}) \times \left(|\mathcal{D}^0|_{-(r-2s),-s;T} + \sum_{j=1}^l T_j^{1/6} |\mathcal{R}^j|_{-(r-2s),-s;\mathcal{T}_j} \right).$$

Proof. We follow the proof of Lemma 5.7 of [7]. For each integer $j \in [1, l]$, let $\mathbf{I}_j^{r,s} := \mathbf{I}_j^{r,s}(m, \mu, T)$ on $\widehat{H}^{-r,-s}$ be orthogonal projection onto $\langle \mathcal{D}_j \rangle$ in the Hilbert space $\widehat{H}_{\mathcal{T}_j}^{-r,-s}$. We get from definitions that

$$\mathcal{D}^j = \mathbf{I}_j^{r,s} (\mathcal{D}^{j-1} + \mathcal{R}^{j-1}) = \mathcal{D}^{j-1} + \mathbf{I}_j^{r,s} (\mathcal{R}^{j-1}).$$

Iteratively applying the triangle inequality, we get

$$\begin{aligned} |\mathcal{D}^l|_{-r,-s} &\leq |\mathcal{D}^{l-1}|_{-r,-s} + |\mathbf{I}_l^{r,s} (\mathcal{R}^{l-1})|_{-r,-s} \\ &\leq |\mathcal{D}^0|_{-r,-s} + \sum_{j=1}^l |\mathbf{I}_{l-j+1}^{r,s} (\mathcal{R}^{l-j})|_{-r,-s} \\ (109) \quad &= |\mathcal{D}^0|_{-r,-s} + \sum_{j=1}^l |\mathbf{I}_j^{r,s} (\mathcal{R}^{j-1})|_{-r,-s}. \end{aligned}$$

By Theorem 4.3 and Theorem 4.9, for any $s > 2$ and any $r \geq 2s$ there exists a constant $C'_{r,s} > 0$ such that

$$\begin{aligned} |\mathcal{D}^0|_{-r,-s} &\leq C'_{r,s} \mathcal{T}_0^{-1/6} (1 + |\lambda m|^{-3s}) |\mathcal{D}^0|_{-(r-2s),-s;\mathcal{T}_0} \\ (110) \quad &= C'_{r,s} T^{-1/6} (1 + |\lambda m|^{-3s}) |\mathcal{D}^0|_{-(r-2s),-s;T}, \end{aligned}$$

and for all integer $j \in [1, l]$, since $\mathbf{I}_j^{r,s} (\mathcal{R}^{j-1}) \in \langle \mathcal{D}^j \rangle$, we have

$$\begin{aligned} |\mathbf{I}_j^{r,s} (\mathcal{R}^{j-1})|_{-r,-s} &\leq C'_{r,s} \mathcal{T}_j^{-1/6} (1 + |\lambda m|^{-3s}) |\mathbf{I}_j^{r,s} (\mathcal{R}^{j-1})|_{-(r-2s),-s;\mathcal{T}_j} \\ (111) \quad &= C'_{r,s} T^{-1/6} (1 + |\lambda m|^{-3s}) T_j^{1/6} |\mathbf{I}_j^{r,s} (\mathcal{R}^{j-1})|_{-(r-2s),-s;\mathcal{T}_j}, \end{aligned}$$

Finally we observe that $\frac{\mathcal{T}_{j-1}}{\mathcal{T}_j} = e^h$, hence there is a constant $C''_{r,s} > 0$ such that

$$|\mathcal{R}^{j-1}|_{-(r-2s),-s;\mathcal{T}_j} \leq C''_{r,s} |\mathcal{R}^{j-1}|_{-(r-2s),-s;\mathcal{T}_{j-1}}.$$

The lemma then follows from the bounds in formulas (109), (110) and (111). \square

Recall from (82) that for all $(x, T) \in M \times \mathbb{R}^+$

$$C_\Gamma(x, T) = \sup_{0 \leq t \leq T} c_\Gamma(x, t).$$

From Lemma 6.17 and Lemma 6.18, we prove

Theorem 6.19. *For every $s > 2$ and $r \geq 5s - 3$, there is a constant $C_{r,s}^{(3)} := C_{r,s}^{(3)}(\Gamma) > 0$ such that for all $T \geq 1$ we have*

$$\begin{aligned} |\mathcal{D}^l|_{-r,-s} &\leq C_{r,s}^{(3)} [C_\Gamma(x, T) + C_\Gamma(h_T(x), T)] \\ &\quad \times (1 + |\lambda m|^{-4s}) T^{-1/6} (1 + \log^{1/2} T). \end{aligned}$$

Proof. Orthogonality shows

$$(112) \quad |\mathcal{D}^0|_{0,-s;T} \leq \left| \int_0^1 (\phi_t^{\lambda,T}(x))^* dt \right|_{0,-s;T}.$$

By Theorem 5.2 and Theorem 6.7, we get a constant $C_s^{(3)} > 0$ such that

$$|\mathcal{D}^0|_{-(r-2s),-s;T} \leq |\mathcal{D}^0|_{0,-s;T} \leq C_s^{(3)} [c_\Gamma(x, T) + c_\Gamma(h_T(x), T)] (1 + \log^{1/2} T).$$

We observe that by the definitions since $\mathcal{T}_j \leq T$ for all $j \in [1, l]$, we have

$$c_\Gamma(x, \mathcal{T}_j) + c_\Gamma(h_T(x), \mathcal{T}_j) \leq C_\Gamma(x, T) + C_\Gamma(h_T(x), T).$$

By Lemma 6.17 and Lemma 6.18, we get

$$\begin{aligned} |\mathcal{D}^l|_{-r,-s} &\leq C_{r,s}^{(3)} (1 + |\lambda m|^{-3s}) [C_\Gamma(x, T) + C_\Gamma(h_T(x), T)] \\ &\quad \times T^{-1/6} (1 + \log^{1/2} T) \left[1 + \left(\frac{1 + |\lambda m|^{-(s-1)}}{|\lambda m|} \right) \sum_{j=1}^{l-1} T_j^{-5/6} \right]. \end{aligned}$$

Since the series converges there exists a constant $C > 0$ such that

$$\left[1 + \left(\frac{1 + |\lambda m|^{-(s-1)}}{|\lambda m|} \right) \sum_{j=1}^{l-1} T_j^{-5/6} \right] \leq C (1 + |\lambda m|^{-4s}),$$

hence the argument is complete. \square

Proof of Theorem 2.3. Let $s > 2$, and let $r \geq 5s - 3$. Set $j = l$ in the orthogonal decomposition (106). Define $\mathcal{R}_{\bar{x},T}^{r,s}$ to be the direct integral in $\widehat{W}^{-r,-s}(M)$ of the distributions $T\mathcal{R}^l$ taken across all irreducible, unitary representations of the group $\mathrm{SL}(2, \mathbb{R}) \times \mathbb{T}$ on $L^2(M \times \mathbb{T})$. Analogously define $\mathcal{D}_{\bar{x},\lambda,T}^{r,s}$ to be the direct integral of the distributions $T^{1/6}\mathcal{D}^l$. By Theorem 6.19 and orthogonality, there is a constant $C_{r,s} := C_{r,s}(\Gamma) > 0$ such that for any $\bar{x} = (x, \theta) \in M \times \mathbb{T}$ and for any $T \geq e$,

$$|\mathcal{D}_{\bar{x},\lambda,T}^{r,s}|_{-r,-s}^2 \leq C_{r,s} [C_\Gamma(x, T) + C_\Gamma(h_T(x), T)]^2 (1 + |\lambda|^{-8s}) (1 + \log T).$$

Similarly, Lemma 6.17 and orthogonality imply

$$|\mathcal{R}_{\bar{x},T}^{r,s}|_{-r,-s}^2 \leq C_{r,s} [C_\Gamma(x, T) + C_\Gamma(h_T(x), T)]^2 \frac{1 + |\lambda|^{-2(s-1)}}{|\lambda|^2} (1 + \log T).$$

These two estimates together give the estimates (16) in Theorem 2.3. This concludes the argument. \square

7. HOROCYCLE MAPS

In this section we prove Theorem 2.7 on the effective equidistribution of horocycle maps. The argument is based on the effective equidistribution of horocycle flows [6] and of twisted horocycle flows (see Theorem 2.3) as well as bounds on solution of the cohomological equation for horocycle maps, which we prove below.

7.1. Cohomological equation. Given $f \in C^\infty(M)$, we obtain Sobolev estimates of solutions to the cohomological equation

$$(113) \quad f = g \circ h_L - g,$$

for $L > 0$. Such estimates have already been obtained in Theorem 1.2 of [23] by studying the horocycle map directly and without reference to the twisted horocycle flow. Here we derive improved estimates from our solution to the cohomological equation of the twisted horocycle flow.

We recall that the horocycle flow is *stable* in the sense that for every non-trivial irreducible representation H_μ and for every function $f \in H_\mu^\infty$ which belongs to the kernel of all horocycle flow invariant distributions, the cohomological equation $Ug = f$ has a unique solution $g \in H_\mu^\infty$ (see Theorem 1.2 of [6]). It is therefore possible to define a Green operator G_μ^U for the horocycle flow defined on the kernel $\text{Ann}(\mathcal{I}_\mu^0) \subset H_\mu^\infty$ of the space \mathcal{I}_μ^0 of all horocycle flow invariant distributions. The Green operator is uniquely defined by the following identity:

$$UG_\mu^U(f) = f \quad \text{for all } f \in \text{Ann}(\mathcal{I}_\mu^0).$$

As the horocycle vector field U , as a linear differential operator on H_μ^∞ is represented in Fourier transform on the space $L_v^2(\mathbb{R})$ by the multiplier $\hat{U} = i\xi$, it follows that the Green operator G_μ^U is represented by the multiplier

$$\hat{G}_\mu^U(\hat{f})(\xi) = \frac{\hat{f}(\xi)}{i\xi}, \quad \text{for all } \hat{f} \in \text{Ann}(\mathcal{I}_\mu^0).$$

For every $\varepsilon > 0$ let $|U|^\varepsilon$ denote the self-adjoint operator defined by the spectral theorem as a function of the skew-adjoint operator U on every unitary representation.

Theorem 7.1. *For all $s \geq 0$ there is a constant $C_s > 0$ such that the following holds. For all irreducible, unitary representations H_μ of the principal or complementary series, for all $\varepsilon \in (0, 1)$, for all $r, a \geq 0$ and for all $f \in H_\mu^\infty \cap \text{Ann}^L(\Gamma)$, there is a unique solution $g \in H_\mu^\infty$ of the cohomological equation (113) for the time- L horocycle map h_L and we have the following estimates:*

$$\|g \circ h_{L/2}\|_{r,s,a} \leq C_s \left(\frac{1+L^{2s}}{L} \|G_\mu^U(f)\|_{r,s,a} + \frac{L^\varepsilon(1+L^s)}{\varepsilon} \|f\|_{r+s,s+1,a+\varepsilon} \right),$$

and, with respect to full Sobolev norms, we have

$$\|g \circ h_{L/2}\|_s \leq C_s \left(\frac{1+L^{2s}}{L} \|G_\mu^U(f)\|_s + \frac{L^\varepsilon(1+L^s)}{\varepsilon} \|f\|_{2s+1+\varepsilon} \right).$$

Proof. By taking the Fourier transform of both sides of equation (113), we get the formula

$$\hat{g}(\xi) = \frac{\hat{f}(\xi)}{e^{iL\xi} - 1}.$$

The estimate will be carried out separately on the intervals $|\xi| \leq \pi/L$, $\xi \geq \pi/L$ and $\xi \leq -\pi/L$. On the bounded interval $I_L := [-\pi/L, \pi/L]$ by the definition of $G_\mu^U(f)$

we can write

$$e^{i\xi L/2} \hat{g}(\xi) = \frac{1}{L} \left(e^{i\xi L/2} \frac{iL\xi}{e^{iL\xi} - 1} \right) \hat{G}_\mu^U(\hat{f}).$$

Observe that the function

$$\phi(\eta) := e^{i\eta/2} \frac{i\eta}{e^{i\eta} - 1}$$

is infinitely differentiable and bounded on $[-\pi, \pi]$, and note $L\xi \in [-\pi, \pi]$ whenever $\xi \in I_L$. Let $\alpha, \beta \in \mathbb{N}$. For $0 \leq \ell \leq \alpha$ and $i + m \leq \beta$, let

$$\phi_{i,m}^{(\ell)}(\xi) := \left(\frac{d}{d\xi} \right)^m \hat{V}^i (2\xi \frac{d}{d\xi})^{\alpha-\ell} \phi(L\xi).$$

There exists a constant $K_{\alpha,\beta}^{(0)} > 0$ such that

$$\|\phi_{i,m}^{(\ell)}\|_{L^\infty(I_L)} \leq K_{\alpha,\beta}^{(0)} (1 + |\nu|)^i L^{m+2i+\alpha-\ell}.$$

By formula (44) we have universal coefficients $(a_\ell^{(\alpha)})(b_{ijk}^{(\beta)})$ such that

$$\begin{aligned} (\hat{V}^\beta \hat{X}^\alpha)[e^{i\xi L/2} \hat{g}(\xi)] &= -\frac{1}{L} \sum_{\ell \leq \alpha} \sum_{\substack{i+j+m \leq \beta \\ k \leq m}} a_\ell^{(\alpha)} b_{ijk}^{(\beta)} \phi_{i,m}^{(\ell)}(\xi) \\ &\quad \times [(\hat{X} - (1 - \nu))^k \hat{V}^j \hat{X}^\ell G_\mu^U(f)](\xi). \end{aligned}$$

By the triangle inequality, it follows that there exists a constant $K_{\alpha,\beta}^{(1)} > 0$ such that

$$\begin{aligned} \|\hat{V}^\beta \hat{X}^\alpha \hat{g}\|_{L^2(I_L)} &\leq K_{\alpha,\beta}^{(1)} L^{-1} (1 + L^{\alpha+2\beta}) \\ (114) \quad &\quad \times \sum_{i+j+k \leq \alpha+\beta} (1 + |\nu|)^i \|\hat{V}^j \hat{X}^k G_\mu^U(f)\|_{L_V^2(\mathbb{R})}. \end{aligned}$$

On the half-lines $\xi \geq \pi/L$ and $\xi \leq -\pi/L$ we proceed in a different way. By a formula in complex analysis (see for instance Chap. V, §4 in [4]) we have

$$\frac{1}{\sin(\xi L/2)} = \frac{1}{\pi} \left(\frac{2\pi}{\xi L} + \sum_{n \geq 1} (-1)^n \frac{\xi L/\pi}{(\xi L/2\pi)^2 - n^2} \right).$$

Hence, we can write

$$ie^{i\xi L/2} \hat{g}(\xi) = \hat{f}(\xi) \frac{2}{\xi L} + \frac{1}{\pi} \sum_{n \geq 1} \frac{4\pi(-1)^n}{L} \hat{f}(\xi) \frac{\xi}{\xi^2 - (2\pi n/L)^2}.$$

For all $\alpha, \beta \geq 0$, we will estimate the $L_V^2(\mathbb{R})$ norm of

$$\hat{V}^\beta \hat{X}^\alpha (ie^{i\xi L/2} \hat{g}).$$

Let $\varepsilon \in (0, 1)$. For $\xi \geq 0$ we can write

$$(115) \quad ie^{i\xi L/2} \hat{g}(\xi) = \frac{2}{L} \frac{\hat{f}(\xi)}{\xi} + \frac{4}{L} \sum_{n \geq 1} (-1)^n \left(\frac{\xi^{1-\varepsilon}}{\xi + 2\pi n/L} \right) \left(\frac{\xi^\varepsilon \hat{f}(\xi)}{\xi - 2\pi n/L} \right).$$

For $0 \leq \ell \leq \alpha$ and $i + m \leq \beta$, let

$$(116) \quad \phi_{i,m,n}^{(\ell),\varepsilon}(\xi) := \left(\frac{d}{d\xi}\right)^m \hat{V}^i (2\xi \frac{d}{d\xi})^{\alpha-\ell} \left(\frac{\xi^{1-\varepsilon}}{\xi + 2\pi n/L}\right).$$

Then as in formula (44), there exist universal coefficients $(a_\ell^{(\alpha)})$, $(b_{ijkm}^{(\beta)})$ such that

$$(117) \quad \begin{aligned} \hat{V}^\beta \hat{X}^\alpha \left(\frac{\xi^{1-\varepsilon}}{\xi + 2\pi n/L}\right) \left(\frac{\xi^\varepsilon \hat{f}(\xi)}{\xi - 2\pi n/L}\right) &= -i \sum_{\ell \leq \alpha} \sum_{\substack{i+j+m \leq \beta \\ k \leq m}} a_\ell^{(\alpha)} b_{ijkm}^{(\beta)} \\ &\times \phi_{i,m,n}^{(\ell),\varepsilon}(\xi) ((\hat{X} - (1-\nu))^k \hat{V}^j \hat{X}^\ell) \left(\frac{\xi^\varepsilon \hat{f}(\xi)}{\xi - 2\pi n/L}\right). \end{aligned}$$

We prove below a bound for the functions $\phi_{i,m,n}^{(\ell),\varepsilon}$, defined in formula (116).

For $(\ell, i, m) = (\alpha, 0, 0)$, since $\xi \geq 0$ and $2\pi n/L \geq 0$, for all $\varepsilon \in (0, 1)$ we have the estimate

$$(118) \quad \xi + \frac{2\pi n}{L} \geq \xi^{1-\varepsilon} \left(\frac{2\pi n}{L}\right)^\varepsilon.$$

It follows that, for all $\varepsilon > 0$ and for all $\xi \geq 0$ we have

$$(119) \quad \|\phi_{i,0,0}^{(\ell),\varepsilon}\|_{L^\infty(\mathbb{R}^+)} \leq \left(\frac{L}{2\pi n}\right)^\varepsilon.$$

For $(\ell, i, m) \neq (\alpha, 0, 0)$, let

$$(120) \quad \begin{aligned} W_{\ell,i,m} &:= \{(w_0, w_1, w_2) \in \mathbb{N}^2 \times \mathbb{N} \setminus \{0\} \mid w_0 \leq i, \\ &w_1 \leq w_2 \leq w_1 + i + m \leq \alpha - \ell + 2i + m\}. \end{aligned}$$

By induction we prove that there exist universal constants $\{c_w \mid w \in W_{\ell,i,m}\}$ such that

$$(121) \quad \left(\frac{d}{d\xi}\right)^m \hat{V}^i (2\xi \frac{d}{d\xi})^{\alpha-\ell} = \sum_{w \in W_{\ell,i,m}} c_w \nu^{w_0} \xi^{w_1} \frac{d^{w_2}}{d\xi^{w_2}}.$$

For $j \in \mathbb{N}$, we also compute that

$$\frac{d^j}{d\xi^j} \left(\frac{\xi}{\xi + 2\pi n/L}\right) = \frac{2\pi n}{L} \frac{(-1)^{j+1} j!}{(\xi + 2\pi n/L)^{j+1}},$$

hence there exist universal constants $\{c'_{w,k} \mid w \in W_{\ell,i,m}, 0 \leq k \leq w_2\}$ such that

$$\begin{aligned} \phi_{i,m,n}^{(\ell),\varepsilon}(\xi) &= \sum_{w \in W_{\ell,i,m}} c'_{w,w_2} \binom{-\varepsilon}{w_2} \nu^{w_0} \frac{\xi^{w_1-w_2+1-\varepsilon}}{\xi + 2\pi n/L} \\ &+ \sum_{w \in W_{\ell,i,m}} \sum_{k=0}^{w_2-1} c'_{w,k} \binom{-\varepsilon}{k} \nu^{w_0} \frac{n}{L} \frac{\xi^{w_1-\varepsilon}}{\xi^k (\xi + 2\pi n/L)^{w_2-k+1}}. \end{aligned}$$

Therefore, we derive from formulas (118) and from the above formula that there is a constant $K_{\alpha,\beta}^{(2)} > 0$ such that, for all $\varepsilon \in (0, 1)$

$$(122) \quad |\phi_{i,m,n}^{(\ell),\varepsilon}(\xi)| \leq K_{\alpha,\beta}^{(2)} (1 + |\nu|)^i (1 + L^{i+m}) \left(\frac{L}{n}\right)^\varepsilon, \quad \text{for all } \xi \geq \pi/L.$$

By the uniform bounds (119) and (122) on the functions $\phi_{i,m,n}^{(\ell),\varepsilon}$ on the half-line $\mathbb{R}_L^+ := \{\xi | \xi \geq \pi/L\}$, it follows that there is a constant $K_{\alpha,\beta}^{(3)} > 0$ such that

$$\begin{aligned} & \left\| \sum_{\substack{i+j+m \leq \beta \\ \ell \leq \alpha, k \leq m}} a_\ell^{(\alpha)} b_{ijkm}^{(\beta)} \phi_{i,m,n}^{(\ell)} (\hat{X} - (1-\nu))^k \hat{V}^j \hat{X}^\ell \left(\frac{\xi^\varepsilon \hat{f}(\xi)}{\xi - \frac{2\pi n}{L}} \right) \right\|_{L_V^2(\mathbb{R}_L^+)} \\ & \leq K_{\alpha,\beta}^{(3)} \left(\frac{L}{n} \right)^\varepsilon \sum_{\substack{i+j+k \leq \alpha+\beta \\ \ell \leq \alpha}} (1+|\nu|)^i (1+L^{\beta-j}) \|\hat{V}^j \hat{X}^k \left(\frac{\xi^\varepsilon \hat{f}(\xi)}{\xi - \frac{2\pi n}{L}} \right)\|_{L_V^2(\mathbb{R}^+)}. \end{aligned}$$

By formula (117), by the above estimate and by Theorem 3.4 it follows that there exists a constant $K_{\alpha,\beta}^{(4)} > 0$ such that

$$\begin{aligned} & \|\hat{V}^\beta \hat{X}^\alpha \left(\frac{\xi^{1-\varepsilon}}{\xi + 2\pi n/L} \right) \left(\frac{\xi^\varepsilon \hat{f}(\xi)}{\xi - 2\pi n/L} \right)\|_{L_V^2(\mathbb{R}_L^+)} \leq K_{\alpha,\beta}^{(4)} \left(\frac{L}{n} \right)^{1+\varepsilon} (1+L^\beta) \\ & \quad \times (1+|\nu|)^\beta \sum_{i+j+k \leq \alpha+\beta+1} (1+|\nu|)^i \|\hat{V}^j \hat{X}^k |\hat{U}|^\varepsilon f\|_{L_V^2(\mathbb{R}^+)}. \end{aligned}$$

Since $\varepsilon > 0$, there exists a constant $K > 0$ such that

$$\sum_{n \geq 1} \left(\frac{L}{n} \right)^{1+\varepsilon} \leq \frac{K}{\varepsilon} L^{1+\varepsilon},$$

hence by formula (115), by the above estimates and by the triangle inequality, there exists a constant $K_{\alpha,\beta}^{(+)} > 0$ such that

$$\begin{aligned} (123) \quad & \|\hat{V}^\beta \hat{X}^\alpha (e^{i\xi L/2} \hat{g})\|_{L_V^2(\mathbb{R}_L^+)} \leq \frac{2}{L} \|\hat{V}^\beta \hat{X}^\alpha \hat{G}_\mu^U(f)\|_{L_V^2(\mathbb{R}^+)} + \frac{K_{\alpha,\beta}^{(+)}}{\varepsilon} L^\varepsilon \\ & \quad \times (1+L^\beta)(1+|\nu|)^\beta \sum_{i+j+k \leq \alpha+\beta+1} (1+|\nu|)^i \|\hat{V}^j \hat{X}^k |\hat{U}|^\varepsilon f\|_{L_V^2(\mathbb{R}^+)}. \end{aligned}$$

The argument is analogous for $\xi \leq 0$. In this case, write

$$ie^{i\xi L/2} \hat{g}(\xi) = \frac{2}{L} \frac{\hat{f}(\xi)}{\xi} + \frac{4}{L} \sum_{n \geq 1} (-1)^{n+1} \left(\frac{|\xi|^{1-\varepsilon}}{\xi - 2\pi n/L} \right) \left(\frac{|\xi|^\varepsilon \hat{f}(\xi)}{\xi + 2\pi n/L} \right),$$

and proceed as before. We conclude that on $\mathbb{R}_L^- := \{\xi | \xi \leq -\pi/L\}$ we have

$$\begin{aligned} (124) \quad & \|\hat{V}^\beta \hat{X}^\alpha (e^{i\xi L/2} \hat{g})\|_{L_V^2(\mathbb{R}_L^-)} \leq \frac{2}{L} \|\hat{V}^\beta \hat{X}^\alpha \hat{G}_\mu^U(f)\|_{L_V^2(\mathbb{R}^-)} + \frac{K_{\alpha,\beta}^{(-)}}{\varepsilon} L^\varepsilon \\ & \quad \times (1+L^\beta)(1+|\nu|)^\beta \sum_{i+j+k \leq \alpha+\beta+1} |1-\nu|^i \|\hat{V}^j \hat{X}^k |\hat{U}|^\varepsilon f\|_{L_V^2(\mathbb{R}^-)}. \end{aligned}$$

For $r \in \mathbb{N}$, and let $s \in \mathbb{N}$ be an even integer. For $\alpha + \beta \leq s$, it follows that is a constant $C_{r,s}^{(6)} > 0$ such that

$$\begin{aligned} |g \circ h_{L/2}|_{r,s} &\leq C_{r,s}^{(6)} (1 + |\mathbf{v}|)^r \left[\frac{1}{L} |G_\mu^U(f)|_{0,s} \right. \\ &\quad \left. + L^\varepsilon (1 + L^\beta) (1 + |\mathbf{v}|)^s \sum_{k=0}^{s+1} |1 - \mathbf{v}|^k ||U|^\varepsilon f|_{0,s+1-k} \right] \\ &\leq C_{r,s}^{(6)} \left(\frac{1}{L} |G_\mu^U(f)|_{r,s} + L^\varepsilon (1 + L^\beta) ||U|^\varepsilon f|_{r+s,s+1} \right). \end{aligned}$$

This concludes the proof of the first estimate of the theorem, for $r \in \mathbb{N}$ and $s \in 2\mathbb{N}$. The estimate for arbitrary real $r \geq 0$ and $s \geq 0$ follows by interpolation, which concludes the estimate in the case $a = 0$.

Now let $a \in \mathbb{N}$ be an even integer. As $(I - U^2)^{a/2}$ commutes with the horocycle map h_L , the function $(I - U^2)^{a/2} g$ is a solution to the cohomological equation (113) with coboundary $(I - U^2)^{a/2} f$. Hence, the above argument applies, and the first estimate holds for all even $a \in \mathbb{N}$. It then holds for all real exponents $a \geq 0$ by interpolation. The second estimate follows from the first. \square

The case of irreducible unitary representations of the discrete series is completely analogous. We have

Theorem 7.2. *For all $s \geq 0$ there is a constant $C_s > 0$ such that the following holds. For all irreducible, unitary representations H_μ of the mock discrete series or discrete series, for all $\varepsilon \in (0, 1)$, for all $r, a \geq 0$ and for all $f \in H_\mu^\infty \cap \text{Ann}^L(\Gamma)$, there is a unique solution $g \in H_\mu^\infty$ of the cohomological equation (113) for the time- L horocycle map h_L and we have the following estimates:*

$$\|g \circ h_{L/2}\|_{r,s,a} \leq C_s \left(\frac{1 + L^{2s}}{L} \|G_\mu^U(f)\|_{r,s,a} + \frac{L^\varepsilon (1 + L^s)}{\varepsilon} \|f\|_{r+3s,s+1,a+\varepsilon} \right).$$

and, with respect to full Sobolev norms, we have

$$\|g \circ h_{L/2}\|_s \leq C_s \left(\frac{1 + L^{2s}}{L} \|G_\mu^U(f)\|_s + \frac{L^\varepsilon (1 + L^s)}{\varepsilon} \|f\|_{4s+1+\varepsilon} \right).$$

Proof. The argument follows the proof of Theorem 7.1. This time the Fourier transform is defined on \mathbb{R}^+ , and the estimates for the twisted cohomological equation are derived from Theorem 3.18 instead of Theorem 3.4. This accounts for the worse loss of derivatives we have for solutions of the cohomological equation in the discrete series compared to the principal and complementary series (compare the statements of Theorem 7.1 and Theorem 7.2). \square

7.2. Effective equidistribution. We recall that H_μ denotes an irreducible, unitary representation space of $L^2(M)$ with Casimir parameter $\mu \in \text{spec}(\square)$. For each $k \in \mathbb{Z}/\{0\}$ and $L > 0$, let $\mathcal{D}_{\mu,k,L}$ be the distribution defined on smooth functions by

$$(125) \quad \mathcal{D}_{\mu,L,k}(f) = D_{k,\mu}^{2\pi/L}(e_k \otimes f).$$

It follows from Lemma 3.3 and Lemma 3.14 that $\mathcal{D}_{\mu,L,k} \in \widehat{H}_\mu^{-(1/2+)}$. By Theorem 1.1 of [6], $H_\mu^{-\infty}$ contains a normalized basis of at most two invariant distributions for the horocycle flow, which can be taken to be generalized eigendistributions for the geodesic flow. Of course, these basis elements are also invariant under the map h_L , and we denote them by \mathcal{D}_μ^+ and \mathcal{D}_μ^- .

Observe that for any $(x, N) \in M \times \mathbb{N}$, the ergodic sum $\frac{1}{N} \sum_{n=0}^{N-1} (h_{nL}(x))^*$ is a measure. Then for any $s > 1$, $r \geq 0$, $a > 1/2$, for any $L > 0$ and for any $(x, N) \in M \times \mathbb{N}$, there exist a horocycle flow-invariant distribution $\mathcal{D}_{x,N,L,r,s,a}^0$ an h_L -invariant distribution $\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}$ and a remainder distribution $\mathcal{R}_{x,N,L,r,s,a}$ such that the following (orthogonal) decomposition holds in the Sobolev space $W^{-r,-s,-a}(M)$:

$$(126) \quad \sum_{n=0}^{N-1} (h_{nL}(x))^* = (\mathcal{D}_{x,N,L,r,s,a}^0 + \mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}) + \mathcal{R}_{x,N,L,r,s,a}.$$

In addition, since each $\mathcal{D}_{\mu,L,k} \in \widehat{H}_\mu^{-(1/2+)}$, the following decompositions hold: there are a sequence of complex numbers $\left\{ c_{\mathcal{D}_{\mu,k,L}}(x, N, L, r, s, a) \right\}_{\mu \in \text{spec}(\square)}$ such that

$$\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}} := \bigoplus_{\mu \in \text{spec}(\square)} \sum_{k \in \mathbb{Z} - \{0\}} c_{\mathcal{D}_{\mu,k,L}}(x, N, L, r, s, a) \mathcal{D}_{\mu,k,L},$$

and a sequence of complex numbers $\left\{ c_{\mathcal{D}_{\mu,L}^\pm}(x, N, L, r, s, a) \right\}_{\mu \in \text{spec}(\square)}$ such that

$$\mathcal{D}_{x,N,L,r,s,a}^0 := \bigoplus_{\mu \in \sigma_{pp}} c_{\mathcal{D}_\mu^+}^s(x, N, L, r, s, a) \mathcal{D}_{\mu,L}^+ + c_{\mathcal{D}_\mu^-}^s(x, N, L, r, s, a) \mathcal{D}_{\mu,L}^-.$$

Lemma 7.6 of [23] and (24) show that for sufficiently large r, s and $a \geq 0$, the distribution $\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}$ is given by a convergent series in each irreducible Sobolev subspace of $W^{-r,-s,-a}(M)$.

In what follows, we estimate the distributions in the decomposition (126). The first step is to derive Sobolev estimates for $\mathcal{R}_{x,N,L,r,s,a}$ from estimates for the solutions of the cohomological equation of horocycle maps proved above.

Recall that for all $(x, L, N) \in M \times \mathbb{R}^+ \times \mathbb{N}$, we set

$$(127) \quad D_\Gamma(x, L, N) := e^{d_M(h_{-L/2}(x))} + e^{d_M(h_{L(N-1/2)}(x))}.$$

Lemma 7.3. *For any $r > 3$, $s > 2$, $a > 2$ and $\varepsilon \in (0, 1)$, there is a constant $C_{r,s,a,\varepsilon} := C_{r,s,a,\varepsilon}(\Gamma) > 0$ such that*

$$\|\mathcal{R}_{x,N,L,r,s,a}\|_{-r,-s,-a} \leq C_{r,s,a,\varepsilon} D_\Gamma(x, L, N) \frac{1 + L^{2+\varepsilon}}{L}.$$

Proof. Let H_μ be an irreducible, unitary representation of $\text{SL}(2, \mathbb{R})$. By Theorem 7.1 and Theorem 7.2, for any function $f \in \text{Ann}^L(\Gamma) \cap H_\mu^\infty$ there is a unique solution $g \in H_\mu^\infty$ of the cohomological equation (113) for the time- L horocycle

map, and for any $\sigma \geq 0$ the function g satisfies the estimate

$$(128) \quad \begin{aligned} \|g \circ h_{L/2}\|_{0,\sigma,0} &\leq C_\sigma \frac{1+L^{2\sigma}}{L} \|G_\mu^U(f)\|_{0,\sigma,0} \\ &\quad + C_\sigma \frac{L^\varepsilon(1+L^\sigma)}{\varepsilon} \|f\|_{3\sigma,\sigma+1,\varepsilon}. \end{aligned}$$

Because $f \in H_\mu^\infty$ is a smooth coboundary, we get as in Lemma 6.17 that

$$\begin{aligned} |\mathcal{R}_{x,N,r,s,a}(f)| &\leq |g \circ h_{NL}(x)| + |g(x)| \\ &\leq |g \circ h_{L/2}(h_{L(N-1/2)}(x))| + |g \circ h_{L/2}(h_{-L/2}(x))|. \end{aligned}$$

By Theorem 5.3 and by Lemma 6.4, for all $\sigma > 1$ there exists a constant $C_{\sigma,\Gamma} > 0$ such that

$$(129) \quad |\mathcal{R}_{x,N,r,s,a}(f)| \leq C_{\sigma,\Gamma} D_\Gamma(x, L, N) (\|Ug \circ h_{L/2}\|_{0,\sigma,0} + \|g \circ h_{L/2}\|_{0,\sigma,0})$$

Notice that Ug is a solution to the cohomological equation (113) for the time- L horocycle map with coboundary Uf , hence by Theorem 7.1 and Theorem 7.2, we get a constant $C'_\sigma > 0$ such that for all $\varepsilon \in (0, 1)$,

$$(130) \quad \begin{aligned} \|Ug \circ h_{L/2}\|_{0,\sigma,0} &\leq C'_\sigma \frac{1+L^{2\sigma}}{L} \|G_\mu^U(Uf)\|_{0,\sigma,0} \\ &\quad + 2C'_\sigma \frac{L^{\varepsilon/2}(1+L^\sigma)}{\varepsilon} \|f\|_{3\sigma,\sigma+1,1+\varepsilon/2}. \end{aligned}$$

By definition, $UG_\mu^U(Uf) = Uf$, hence by uniqueness of solutions for the cohomological equation of the horocycle flow, (see Theorem 1.2 of [6] or the proof of Theorem 3.4), we have $G_\mu^U(Uf) = f$. By the bounds in formulas (128), (129) and (130) it then follows that

$$(131) \quad \begin{aligned} |\mathcal{R}_{x,N,r,s,a}(f)| &\leq C'_{\sigma,\Gamma} D_\Gamma(x, L, N) \frac{1+L^{2\sigma}}{L} (\|f\|_{0,\sigma,0} + \|G_\mu^U(f)\|_{0,\sigma,0}) \\ &\quad + 2C'_{\sigma,\Gamma} D_\Gamma(x, L, N) \frac{L^{\varepsilon/2}(1+L^\sigma)}{\varepsilon} \|f\|_{3\sigma,\sigma+1,1+\varepsilon/2}. \end{aligned}$$

Now observe that Theorem 1.2 of [6] shows that for all $\varepsilon' > 0$, there is a constant $C_{\sigma,\varepsilon'} := C_{\sigma,\varepsilon'}(\Gamma) \geq 1$ such that

$$\|G_\mu^U(f)\|_{0,\sigma,0} \leq \|G_\mu^U(f)\|_\sigma \leq C_{\sigma,\varepsilon'} \|f\|_{\sigma+1+\varepsilon'} \leq C_{\sigma,\varepsilon'} \|f\|_{0,\sigma+1+\varepsilon',\sigma+1+\varepsilon'}.$$

Taking $\sigma, \varepsilon, \varepsilon'$ so that $r > 3\sigma$, $s, a > \sigma + 1 + \varepsilon'$ and $2\sigma < 2 + \varepsilon$, we get that

$$|\mathcal{R}_{x,N,r,s,a}(f)| \leq C_{r,s,a,\varepsilon} D_\Gamma(x, L, N) \frac{1+L^{2+\varepsilon}}{L} \|f\|_{r,s,a}.$$

This proves Lemma 7.3 for coboundaries in each irreducible, unitary representation of $\mathrm{SL}(2, \mathbb{R})$, and the general statement now follows by orthogonality of the decomposition of $W^{r,s,a}(M)$ into irreducible components and of every function $f \in W^{r,s,a}(M)$ into a coboundary component and an orthogonal component. \square

For each $k \in \mathbb{Z}/\{0\}$, let

$$\mathcal{D}_{L,k} := \bigoplus_{\mu \in \text{spec}(\square)} c_{\mathcal{D}_{\mu,L,k}}(x, N, L, r, s, a) \mathcal{D}_{\mu,L,k}.$$

By applying $\frac{1}{L} \int_0^L e^{2\pi i \frac{k}{L} t} (h_t(x))^* dt$ or $\frac{1}{L} \int_0^L (h_t(x))^* dt$ to the ergodic sum, Lemma 7.2 of [23] and the definition of $\mathcal{D}_{L,k}$ and \mathcal{D}^0 give

Lemma 7.4. *For all $(x, N, L) \in M \times \mathbb{Z}^+ \times \mathbb{R}^+$ and $k \in \mathbb{Z} \setminus \{0\}$, we have the following distributional identities in $\mathcal{E}'(M)$:*

$$\begin{aligned} \mathcal{D}_{L,k} &= \frac{1}{L} \int_0^{NL} e^{2\pi i t k/L} (h_t(x))^* dt - \frac{1}{L} \int_0^L e^{2\pi i t k/L} h_{-t} \mathcal{R}_{x,N,L,r,s,a} dt; \\ \mathcal{D}^0 &= \frac{1}{L} \int_0^{NL} (h_t(x))^* dt - \frac{1}{L} \int_0^L h_{-t} \mathcal{R}_{x,N,L,r,s,a} dt. \end{aligned}$$

As noted, Lemma 7.6 of [23] and (24) show that for sufficiently large r, s and a , the distribution $\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}$ is given by a convergent series in each irreducible component of $W^{-r,-s,-a}(M)$. For such (r, s, a) , we have

$$(132) \quad \mathcal{D}_{x,N,L,r,s,a}^{\text{twist}} = \sum_{k \in \mathbb{Z}/\{0\}} \mathcal{D}_{L,k}.$$

The distribution $\mathcal{R}_{x,N,L,r,s,a}$ is controlled by Lemma 7.3. We now estimate the integral of $\mathcal{R}_{x,N,L,r,s,a}$ along the horocycle flow.

Lemma 7.5. *For all $r > 3$, $s > 2$, $a > 2$ and $\varepsilon \in (0, 1)$ there exists a constant $C'_{r,s,a,\varepsilon} > 0$ such that*

$$\begin{aligned} \frac{1}{L} \int_0^L |\mathcal{R}_{x,N,L,r,s,a}(f \circ h_t)| dt &\leq C'_{r,s,a,\varepsilon} D_\Gamma(x, L, N) \\ &\quad \times \frac{1 + L^{2+\varepsilon}}{L} (1 + L^{2s}) \|f\|_{r,s,s+a}. \end{aligned}$$

Proof. By Lemma 7.3, for all $r > 3$, $s > 2$, $a > 2$ and $\varepsilon \in (0, 1)$ there exists a constant $C_{r,s,a,\varepsilon} > 0$ such that, for all $f \in C^\infty(M)$, we have

$$\begin{aligned} \frac{1}{L} \int_0^L |\mathcal{R}_{x,N,L,r,s,a}(f \circ h_t)| dt &\leq C_{r,s,a,\varepsilon} D_\Gamma(x, L, N) \\ &\quad \times \frac{1 + L^{2+\varepsilon}}{L^2} \int_0^L \|f \circ h_t\|_{r,s,a} dt. \end{aligned}$$

Also notice that for any $f \in H_\mu^\infty$,

$$\begin{aligned} U(f \circ h_t) &= (Uf) \circ h_t, \\ X(f \circ h_t) &= [(X + tU)f] \circ h_t, \\ (133) \quad V(f \circ h_t) &= [(V - 2tX - t^2U)f] \circ h_t. \end{aligned}$$

Hence, for all $r, s, a \in \mathbb{N}$, there is a constant $C_s > 0$ such that

$$\|f \circ h_t\|_{r,s,a} \leq C_s (1 + t^{2s}) \|f\|_{r,s,s+a}.$$

By interpolation, this estimate holds for all $r, s, a \geq 0$, hence the lemma follows. \square

Next we estimate the Sobolev norms of the distribution $\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}$. For all $(x, L, N) \in M \times \mathbb{R}^+ \times N$, let $C_\Gamma(x, L, N)$ denote the positive constant defined by the formula

$$(134) \quad C_\Gamma(x, L, N) := C_\Gamma(x, NL) + C_\Gamma(h_{NL}(x), NL).$$

Lemma 7.6. *Let $s > 2$, $a > 2$ and $r > 5s - 3$. For all $\varepsilon > 0$, there exists a constant $C_{r,s,a,\varepsilon}^{(2)} > 0$ and for $(x, N) \in M \times N$ and $L > 0$ there exists a decomposition*

$$\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}} = \tilde{\mathcal{D}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}} + \tilde{\mathcal{R}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}}$$

such that the following estimates hold:

$$\begin{aligned} \|\tilde{\mathcal{D}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}}\|_{-r,-s,-(1+\varepsilon)} &\leq C_{r,s,a,\varepsilon}^{(2)} C_\Gamma(x, L, N) \\ &\quad \times (1 + L^{8s+\varepsilon})(NL)^{5/6} \log^{1/2}(e + NL); \\ \|\tilde{\mathcal{R}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}}\|_{-r,-s,-(s+a+1+\varepsilon)} &\leq C_{r,s,a,\varepsilon}^{(2)} D_\Gamma(x, L, N)(1 + L^{2s+2+\varepsilon}). \end{aligned}$$

Proof. Notice that for any $\mu \in \text{spec}(\square)$, for any $L > 0$, $k \in \mathbb{Z}/\{0\}$ and for any $\varepsilon \in \mathbb{R}^+$, by the definition of the invariant distributions $\mathcal{D}_{\mu,L,k}$ we have

$$(135) \quad \mathcal{D}_{\mu,L,k} = (1 + \frac{4\pi^2 k^2}{L^2})^{-\varepsilon/2} (I - U^2)^{\varepsilon/2} \mathcal{D}_{\mu,L,k}.$$

Then by Lemma 7.4 we derive the formula

$$(136) \quad \begin{aligned} \mathcal{D}_{L,k} &= (1 + \frac{4\pi^2 k^2}{L^2})^{-\varepsilon/2} (I - U^2)^{\varepsilon/2} \left(\frac{1}{L} \int_0^{NL} e^{2\pi i k t/L} (h_t(x))^* dt \right) \\ &\quad - (1 + \frac{4\pi^2 k^2}{L^2})^{-\varepsilon/2} (I - U^2)^{\varepsilon/2} \left(\frac{1}{L} \int_0^L e^{2\pi i k t/L} h_{-t} \mathcal{R}_{x,N,L,r,s,a} dt \right). \end{aligned}$$

Then integration by parts shows

$$\begin{aligned} &\frac{1}{L} \int_0^{NL} e^{2\pi i k t/L} (I - U^2)^{\varepsilon/2} f \circ h_t(x) dt \\ &= \frac{1}{2\pi i k} [(I - U^2)^{\varepsilon/2} f \circ h_{NL}(x) - (I - U^2)^{\varepsilon/2} f(x)] \\ &\quad - \frac{1}{2\pi i k} \int_0^{NL} e^{2\pi i k t/L} U (I - U^2)^{\varepsilon/2} f \circ h_t(x) dt. \end{aligned}$$

By Theorem 2.3, for all $s > 2$, for $r > 5s - 3$, there is a constant $C_{r,s} > 0$ such that, for all $\varepsilon \in \mathbb{R}^+$ we have

$$(137) \quad \begin{aligned} &\|(I - U^2)^{\varepsilon/2} \left(\frac{1}{L} \int_0^{NL} e^{2\pi i k t/L} (h_t(x))^* dt \right)\|_{-r,-s,-(1+\varepsilon)} \\ &\leq \frac{C_{r,s}}{|k|} (1 + L^{8s}) C_\Gamma(x, L, N) (NL)^{5/6} \log^{1/2}(e + |NL|). \end{aligned}$$

Now we estimate the integral of the remainder distribution from (136). Integration by parts gives

$$\begin{aligned} & \left| \frac{1}{L} \int_0^L e^{2\pi i t k/L} \mathcal{R}_{x,N,L,r,s,a}((I-U^2)^{\varepsilon/2} f \circ h_t) dt \right| \\ & \leq \frac{1}{2\pi i k} \left| \mathcal{R}_{x,N,L,r,s,a}((I-U^2)^{\varepsilon/2} f \circ h_L) - \mathcal{R}_{x,N,L,r,s,a}((I-U^2)^{\varepsilon/2} f) \right| \\ & \quad + \frac{1}{2\pi i k} \left| \int_0^L e^{2\pi i t k/L} \mathcal{R}_{x,N,L,r,s,a}(U(I-U^2)^{\varepsilon/2} f) \circ h_t dt \right|. \end{aligned}$$

Then Lemma 7.5 shows that for all $r > 3$, $s > 2$, $a > 2$ and $\varepsilon' \in (0, 1)$, there is a constant $C_{r,s,a,\varepsilon'} > 0$ such that

$$\begin{aligned} & \left\| (I-U^2)^{\varepsilon/2} \left(\frac{1}{L} \int_0^L e^{2\pi i t k/L} h_{-t} \mathcal{R}_{x,N,L,r,s,a} dt \right) \right\|_{-r,-s,-(s+a+1+\varepsilon)} \\ (138) \quad & \leq \frac{C_{r,s,a,\varepsilon'}}{|k|} D_{\Gamma}(x, L, N) (1 + L^{2s+2+\varepsilon'}). \end{aligned}$$

A similar estimate holds for the finite factor.

Now define

$$\begin{aligned} \tilde{\mathcal{D}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}} &:= \sum_{k \in \mathbb{Z}/\{0\}} \left(1 + \frac{4\pi^2 k^2}{L^2} \right)^{-\varepsilon/2} \\ (139) \quad & \times (I-U^2)^{\varepsilon/2} \left(\frac{1}{L} \int_0^{NL} e^{2\pi i k t/L} (h_t(x))^* dt \right) \\ \tilde{\mathcal{R}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}} &:= \sum_{k \in \mathbb{Z}/\{0\}} \left(1 + \frac{4\pi^2 k^2}{L^2} \right)^{-\varepsilon/2} \\ & \times (I-U^2)^{\varepsilon/2} \left(\frac{1}{L} \int_0^L e^{2\pi i t k/L} h_{-t} \mathcal{R}_{x,N,L,r,s,a} dt \right). \end{aligned}$$

By construction and by formula 136, we have that

$$\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}} = \tilde{\mathcal{D}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}} + \tilde{\mathcal{R}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}}.$$

In addition, by the estimate in formula (137), for any $\varepsilon > 0$, there exists a constant $C_{r,s,\varepsilon} > 0$ such that

$$\begin{aligned} & \left\| \tilde{\mathcal{D}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}} \right\|_{-r,-s,-(1+\varepsilon)} \leq C_{r,s,\varepsilon} (1 + L^{8s+\varepsilon}) \\ & \quad \times C_{\Gamma}(x, L, N) (NL)^{5/6} \log^{1/2}(e + NL). \end{aligned}$$

By the estimate in formula (138), there exists a constant $C_{r,s,a,\varepsilon} > 0$ such that

$$\left\| \tilde{\mathcal{R}}_{x,N,L,r,s,a,\varepsilon}^{\text{twist}} \right\|_{-r,-s,-(s+a+1+\varepsilon)} \leq C_{r,s,a,\varepsilon} D_{\Gamma}(x, L, N) (1 + L^{2s+2+\varepsilon}).$$

□

Proof of Theorem 2.7. By formula (126) the distribution given by the ergodic sum of the time- L horocycle map for a point $x \in M$ and up to time $N \in \mathbb{N}$ can be decomposed into a distribution $\mathcal{D}_{x,N,L,r,s,a}^0$ invariant under the horocycle flow, a distribution $\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}$ invariant under the horocycle map (but not under the horocycle flow) and a remainder distribution $\mathcal{R}_{x,N,L,r,s,a}$.

The estimate for the distribution $\mathcal{D}_{x,N,L,r,s,a}^0$ follows from Lemmas 7.4 and 7.5, and the estimate for the distribution $\mathcal{D}_{x,N,L,r,s,a}^{\text{twist}}$ follows from Lemma 7.6. Finally, the estimate for the remainder term $\mathcal{R}_{x,N,L,r,s,a}$ is given by Lemma 7.3. This concludes the proof of Theorem 2.7. \square

8. A RESULT ON SHAH'S QUESTION

In this section we prove Theorem 1.3. We follow the approach from Theorem 3.1 of the paper [25] by A. Venkatesh where, for $f \in C^\infty(M)$ of zero average, the sum

$$\frac{1}{N} \sum_{n=1}^N f \circ h_{n^{1+\delta}}(x)$$

is controlled by sampling f along suitable arithmetic progressions.

Proof of Theorem 1.3. Let $f \in C^\infty(M)$ such that $\int_M f d\text{vol} = 0$.

By Taylor formula for every fixed $N \in \mathbb{N} \setminus \{0\}$ and for every $t \geq 0$ we have

$$(140) \quad (N+t)^{1+\delta} = N^{1+\delta} + (1+\delta)N^\delta t + O(N^{\delta-1}t^2).$$

Thus the function $(N+t)^{1+\delta}$ is well approximated by its linear Taylor polynomial as long as $N^{\delta-1}t^2$ is small for N large.

Motivated by the above remark we fix $\varepsilon > 0$, we set $N_1 := [N^{1-\varepsilon}] + 1$, and for all $j \in \mathbb{N} \setminus \{0\}$ we define

$$N_{j+1} := N_j + [N_j^{(1-\delta)/2-\varepsilon}].$$

Let $J \in \mathbb{N}$ be such that $N_J \leq N \leq N_{J+1}$. This implies in particular that $N - N_J \leq N_{J+1} - N_J = [N_J^{(1-\delta)/2-\varepsilon}] \leq N^{(1-\delta)/2-\varepsilon}$. Hence, there is a constant $C_f > 0$ such that

$$\begin{aligned} \frac{1}{N} \left| \sum_{n=0}^{N_1-1} f \circ h_{n^{1+\delta}}(x) \right| &\leq C_f N^{-\varepsilon}, \\ \frac{1}{N} \left| \sum_{n=N_J}^{N-1} f \circ h_{n^{1+\delta}}(x) \right| &\leq C_f N^{-(1+\delta)/2-\varepsilon}, \end{aligned}$$

and both terms converge to zero as $N \rightarrow +\infty$. Then we need to estimate

$$\frac{1}{N} \left| \sum_{n=N_1}^{N_J-1} f \circ h_{n^{1+\delta}}(x) \right|.$$

We then let, for all $j \in \{1, \dots, J-1\}$,

$$(141) \quad L_j := (1+\delta)N_j^\delta.$$

By the triangular inequality we have

$$(142) \quad \begin{aligned} \frac{1}{N} \left| \sum_{n=N_1}^{N_J-1} f \circ h_{n^{1+\delta}}(x) \right| &\leq \frac{1}{N} \left| \sum_{j=1}^{J-1} \sum_{k=0}^{[N_j^{1-\delta-\varepsilon}]-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) \right| \\ &+ \frac{1}{N} \left| \sum_{n=N_1}^{N_J-1} f \circ h_{n^{1+\delta}}(x) - \sum_{j=1}^{J-1} \sum_{k=0}^{[N_j^{1-\delta-\varepsilon}]-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) \right|. \end{aligned}$$

We begin by estimating the first term on the RHS in the above inequality which is composed of several sums along arithmetic progressions.

By Theorem 1.2 and by the effective equidistribution of the horocycle flow (see Theorem 1.5 in [6]), for every $\varepsilon > 0$ there exists a constant $C_{f,\varepsilon} > 0$ such that, for all $j \in \{1, \dots, J-1\}$ and for all $x \in M$, we have

$$(143) \quad \begin{aligned} \left| \sum_{k=0}^{[N_j^{(1-\delta)/2-\varepsilon}]-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) \right| &\leq C_{f,\varepsilon} L_j^{-1} (L_j N_j^{(1-\delta)/2-\varepsilon})^{1-\delta_{\mu_0}^-} \\ &+ C_{f,\varepsilon} \left(L_j^{1/6+\varepsilon} (L_j N_j^{(1-\delta)/2-\varepsilon})^{5/6} (\log N_j)^{1/2} + L_j^{5+\varepsilon} \right). \end{aligned}$$

By these inequalities, since $1 - \delta_{\mu_0}^- < 1$, under the hypothesis that $\delta < 1/13$, it follows from the estimate in formula (143) that there exists $\varepsilon > 0$ (sufficiently small) such that for all $x \in M$ we have

$$\left| \sum_{k=0}^{[N_j^{(1-\delta)/2-\varepsilon}]-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) \right| \leq C_{f,\varepsilon} [N_j^{(1-\delta)/2-\varepsilon}] N_j^{-\varepsilon}.$$

Since by construction we have that $[N_j^{(1-\delta)/2-\varepsilon}] = N_{j+1} - N_j$ and $N_j \geq N_1 = [N^{1-\varepsilon}] + 1$ for all $j \in \{1, \dots, J-1\}$, and also $N_J \leq N$, by telescopic summation it then follows that for all $x \in M$ we have

$$\frac{1}{N} \left| \sum_{j=1}^{J-1} \sum_{k=0}^{[N_j^{(1-\delta)/2-\varepsilon}]-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) \right| \leq C_{f,\varepsilon} \frac{N_J}{N} N^{-\varepsilon(1-\varepsilon)} \leq C_{f,\varepsilon} N^{-\varepsilon(1-\varepsilon)}.$$

In particular we have proved that, uniformly over $x \in M$,

$$(144) \quad \lim_{N \rightarrow +\infty} \frac{1}{N} \sum_{j=1}^{J-1} \sum_{k=0}^{[N_j^{(1-\delta)/2-\varepsilon}]-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) = 0.$$

Now we estimate the second term on the RHS of formula (142), that is

$$\frac{1}{N} \left| \sum_{n=N_1}^{N_J-1} f \circ h_{n^{1+\delta}}(x) - \sum_{j=1}^{J-1} \sum_{k=0}^{N_{j+1}-N_j-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) \right|.$$

By Taylor formula (140), for $0 \leq k \leq [N_j^{(1-\delta)/2-\varepsilon}] - 1$, we have

$$(N_j + k)^{1+\delta} - (N_j^{1+\delta} + kL_j) = O(N_j^{-2\varepsilon}),$$

hence there is a constant $C_f > 0$ such that, for all $0 \leq k < N_{j+1} - N_j$, we have

$$|f \circ h_{(N_j+k)^{1+\delta}}(x) - f \circ h_{N_j^{1+\delta}+kL_j}(x)| \leq C_f N_j^{-2\varepsilon}.$$

It follows that there is a constant $C_f > 0$ such that

$$\begin{aligned} & \left| \sum_{n=N_1}^{N_j-1} f \circ h_{n^{1+\delta}}(x) - \sum_{j=1}^{J-1} \sum_{k=0}^{N_{j+1}-N_j-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) \right| \\ & \leq \sum_{j=1}^{J-1} \sum_{k=0}^{N_{j+1}-N_j-1} |f \circ h_{(N_j+k)^{1+\delta}}(x) - f \circ h_{N_j^{1+\delta}+kL_j}(x)| \\ & \leq C_f \sum_{j=1}^{J-1} (N_{j+1} - N_j) N_j^{-2\varepsilon} \leq C_f N N_1^{-2\varepsilon} \leq C_f N N^{-2\varepsilon(1-\varepsilon)}. \end{aligned}$$

Thus, we have that, uniformly over $x \in M$,

$$\lim_{N \rightarrow +\infty} \frac{1}{N} \left(\sum_{n=N_1}^{N-1} f \circ h_{n^{1+\delta}}(x) - \sum_{j=1}^{J-1} \sum_{k=0}^{N_j-1} f \circ h_{N_j^{1+\delta}+kL_j}(x) \right) = 0.$$

Theorem 1.3 follows from this, formulas (142) and (144). \square

APPENDIX A.

A.1. Line and upper half-plane models of $SL(2, \mathbb{R})$. The irreducible representation spaces for $SL(2, \mathbb{R}) \times \mathbb{T}$ can be studied in concrete, unitarily equivalent models. We presently describe the line and upper half-plane models for $SL(2, \mathbb{R})$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R})$. Let $\mu \in \text{spec}(\square)$ be a Casimir parameter, and let H_μ be an irreducible, unitary representation space in the kernel of $(\mu - \square)$. Let $\nu = \sqrt{1 - \mu}$ be a representation parameter. We denote by H_μ the following models for the principal and complementary series representation spaces. In the first model (the line model) the Hilbert space is a space of functions on \mathbb{R} with the following norms. If $\mu \geq 1$, then $\nu \in i\mathbb{R}$ and $\|f\|_0 = \|f\|_{L^2(\mathbb{R})}$. If $0 < \mu < 1$, then $0 < \nu < 1$ and

$$\|f\|_{H_\mu} = \left(\int_{\mathbb{R}^2} \frac{f(x)\overline{f(y)}}{|x-y|^{1-\nu}} dx dy \right)^{1/2}.$$

The group action is defined by

$$\pi_\nu : SL(2, \mathbb{R}) \rightarrow \mathcal{U}(H)$$

$$\pi_\nu(A)f(x) = |-cx + a|^{-(\nu+1)} f\left(\frac{dx-b}{-cx+a}\right),$$

where $x \in \mathbb{R}$.

The vector fields for the model H_μ on \mathbb{R} are

$$\begin{aligned} X &= -(1 + \nu) - 2x \frac{\partial}{\partial x}; \\ U &= -\frac{\partial}{\partial x}; \\ V &= (1 + \nu)x + x^2 \frac{\partial}{\partial x}. \end{aligned}$$

For $\mu \leq 0$, we let \mathbb{H} be the upper half-plane. The upper half-plane model is also denoted by H_μ , where now $\mu \in \{-n^2 + 2n : n \in \mathbb{Z}^+\}$, and its norm is

$$\|f\|_{H_\mu} = \begin{cases} \left(\int_{\mathbb{H}} |f(x+iy)|^2 y^{n-2} dx dy \right)^{1/2}, & n \geq 2; \\ \left(\sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx \right)^{1/2}, & n = 1. \end{cases}$$

This model has the group action $\pi_n : SL(2, \mathbb{R}) \rightarrow \mathcal{U}(H_\mu)$ defined by

$$\pi_n(A) : f(z) \rightarrow (-cz + a)^{-n} f\left(\frac{dz - b}{-cz + a}\right).$$

The anti-holomorphic discrete series is similar, but we only consider the holomorphic case because there is a complex anti-linear isomorphism between two series of the same Casimir parameter.

Then the vector fields in the model H are:

$$\begin{aligned} X &= -(1 + \nu) - 2z \frac{\partial}{\partial z} \\ U &= -\frac{\partial}{\partial z} \\ V &= (1 + \nu)z + z^2 \frac{\partial}{\partial z}. \end{aligned}$$

APPENDIX B.

Proof of Lemma 3.1. If $\mu \geq 1$, then Lemma 3.1 is immediate. So say $0 < \mu < 1$. Then

$$(145) \quad \|f\|_0^2 = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f(x)f(y)}{|x-y|^{1-\nu}} dx dy = \langle f * K, f \rangle,$$

where $K(x) = |x|^{1-\nu}$. A computation shows that the Fourier transform $\hat{K}(1)$ is defined, and moreover, for any $\xi \in \mathbb{R}$,

$$\hat{K}(\xi) = |\xi|^{-\nu} \hat{K}(1).$$

Because \hat{K} is not identically zero, we have $\hat{K}(1) \neq 0$.

Thus, Plancherel's equality gives

$$(145) = \hat{K}(1) \int_{\mathbb{R}} |\hat{f}(\xi)|^2 |\xi|^{-\nu} d\xi.$$

□

Proof of Lemma 3.16. Let $z = x + iy$. Using Lemma 3.15, we have

$$\begin{aligned} \int_{\mathbb{R}} f(x+iy) e^{-i\xi x} dx &= e^{-\xi y} \int_{\mathbb{R}} f(z) e^{-i\xi z} dx \\ &= e^{-\xi y} \hat{f}^y(\xi) \\ (146) \quad &= e^{-\xi y} \hat{f}(\xi). \end{aligned}$$

Because $\nu \geq 1$, Sobolev embedding shows $f(\cdot + iy) \in L^1(\mathbb{R})$ for any $y \in \mathbb{R}^+$. Moreover, as f is smooth, $\int_{\mathbb{R}} f(x+iy) e^{-i\xi x} dx$ is also in $L^1(\mathbb{R})$. So the Fourier inversion

formula followed by Lemma 3.15 gives

$$\begin{aligned}
 f(z) &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(t+iy) e^{-i\xi t} dt \right) e^{i\xi x} d\xi \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}} \left(e^{\xi y} \int_{\mathbb{R}} f(t+iy) e^{-i\xi t} dt \right) e^{i\xi z} d\xi \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^+} \hat{f}^y(\xi) e^{i\xi z} d\xi \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^+} \hat{f}(\xi) e^{i\xi z} d\xi.
 \end{aligned}$$

We now consider the L^2 norm for $\nu = 0$. The Plancherel theorem and formula (146) give

$$\begin{aligned}
 \|f\|_0^2 &= \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^2 dx \\
 &= \frac{1}{2\pi} \sup_{y>0} \int_{\mathbb{R}^+} \left| \int_{\mathbb{R}} f(x+iy) e^{-i\xi x} dx \right|^2 d\xi \\
 &= \frac{1}{2\pi} \sup_{y>0} \int_{\mathbb{R}^+} e^{-2\xi y} |\hat{f}(\xi)|^2 d\xi \\
 &= \frac{1}{2\pi} \int_{\mathbb{R}^+} |\hat{f}(\xi)|^2 d\xi.
 \end{aligned}$$

For $\nu \geq 1$, we have

$$\begin{aligned}
 \|f\|_0^2 &= \int_0^\infty \int_{\mathbb{R}} |f(x+iy)|^2 y^{\nu-1} dx dy \\
 &= \frac{1}{2\pi} \int_0^\infty \langle e^{-2\xi y} \hat{f}, \hat{f} \rangle_{L^2(\mathbb{R})} y^{\nu-1} dy \\
 (147) \quad &= \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \int_0^\infty y^{\nu-1} e^{-2\xi y} dy d\xi.
 \end{aligned}$$

Using integration by parts $\nu - 1$ times, we conclude

$$\|f\|_0^2 = \frac{(\nu-1)!}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \frac{d\xi}{(2\xi)^\nu}.$$

□

Proof of Lemma 4.5: When H is a principal series representation for G , the operator $U_{\mathcal{T}}$ is unitary, so H is in the complementary series. Recall that the norm for the line models is

$$\|f\|_0 = \left(\int_{\mathbb{R}^2} \frac{f(x) \overline{f(y)}}{|x-y|^{1-\nu}} dx dy \right)^{1/2}.$$

From Lemma 3.1 of [23], the Fourier transform of f is defined and continuous everywhere.

Without loss of generality, assume $\lambda > 0$. We have

$$\|U_{\mathcal{T}}f\|_0^2 = \mathcal{T}^{1/3} \int_{\mathbb{R}} |\hat{f}(\lambda + \mathcal{T}^{1/3}(\xi - \lambda))|^2 |\xi|^{-\nu} d\xi.$$

Let $y - \lambda = \mathcal{T}^{1/3}(\xi - \lambda)$. Then

$$\xi^{-\nu} = \frac{\mathcal{T}^{\nu/3}}{(y + (\mathcal{T}^{1/3} - 1)\lambda)^{\nu}}.$$

Because $y \in (\frac{\lambda}{2}, \frac{3\lambda}{2})$, we get the upper and lower bounds

$$\lambda^{-\nu} \left(\frac{\mathcal{T}^{1/3}}{\mathcal{T}^{1/3} + 1/2} \right)^{\nu} \leq \xi^{-\nu} \leq \left(\frac{\mathcal{T}^{1/3}}{\mathcal{T}^{1/3} - 1/2} \right)^{\nu} \lambda^{-\nu}.$$

These bounds are made largest and smallest by setting $\nu = 1$ and $\mathcal{T} = 1$. We get,

$$\sqrt{\frac{2}{3}} \lambda^{-\nu/2} \|\hat{f}\|_{L^2(\mathbb{R})} \leq \|U_{\mathcal{T}}f\|_0^2 \leq \sqrt{2} \lambda^{-\nu/2} \|\hat{f}\|_{L^2(\mathbb{R})}.$$

We have the same upper and lower bounds for $\|f\|_0$, so

$$\frac{1}{\sqrt{3}} \leq \frac{\|U_{\mathcal{T}}f\|_0}{\|f\|_0} \leq \sqrt{3}.$$

This completes the proof of Lemma 4.5. \square

Proof of Lemma 4.11. In this case, $I_{\lambda} = [\lambda - 1/2, \lambda + 1/2]$. Setting $(-1)! := 1$, Lemma 3.16 gives

$$\|U_{\mathcal{T}}f\|_0^2 = \frac{(\nu - 1)!}{(2\pi)^{\nu}} \mathcal{T}^{1/3} \int_{\mathbb{R}} |\hat{f}(\lambda + \mathcal{T}^{1/3}(\xi - \lambda))|^2 |\xi|^{-\nu} d\xi.$$

So let $y - \lambda = \mathcal{T}^{1/3}(\xi - \lambda)$, which means

$$\xi^{-\nu} = \frac{\mathcal{T}^{\nu/3}}{(y + (\mathcal{T}^{1/3} - 1)\lambda)^{\nu}}.$$

Observe $\xi^{-\nu}$ satisfies

$$\left(\frac{\mathcal{T}^{1/3}}{\lambda \mathcal{T}^{1/3} + 1/2} \right)^{\nu} \leq \xi^{-\nu} \leq \left(\frac{\mathcal{T}^{1/3}}{\lambda \mathcal{T}^{1/3} - 1/2} \right)^{\nu}.$$

These bounds are made worse by setting $\mathcal{T} = 1$, so that we get

$$\frac{(\nu - 1)!}{(2\pi(\lambda + 1/2))^{\nu}} \|\hat{f}\|_{L^2(\mathbb{R})} \leq \|U_{\mathcal{T}}f\|_{\mathcal{H}_{\mu}} \leq \frac{(\nu - 1)!}{(2\pi(\lambda - 1/2))^{\nu}} \|\hat{f}\|_{L^2(\mathbb{R})}.$$

We get the same upper and lower bounds for $\|f\|_0$, so there is a constant $C > 0$ such that

$$\left(\frac{1 - 1/(2\lambda)}{1 + 1/(2\lambda)} \right)^{\nu} \leq \frac{\|U_{\mathcal{T}}f\|_0}{\|f\|_0} \leq \left(\frac{1 + 1/(2\lambda)}{1 - 1/(2\lambda)} \right)^{\nu}.$$

Now because $\lambda \geq \nu + 1$, we get a constant $C > 0$ such that

$$\frac{1}{C} \leq \frac{\|U_{\mathcal{T}}f\|_0}{\|f\|_0} \leq C.$$

□

APPENDIX C.

Proof of Lemma 6.4. Let $\mathbb{H} := \{w = x + iy \in \mathbb{C} : \text{Im}(w) > 0\}$ be the Poincaré upper-half plane endowed with the Riemannian hyperbolic metric

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y},$$

and let M be the unit tangent bundle of a surface $S = \Gamma \backslash \mathbb{H}$.

When M is compact, the horocycle flow has no periodic orbits and all orbits are transverse to the X - V leaves. By compactness, there is a constant $C'_\Gamma > 0$ such that for any $x \in M$ α_x is injective on the domain

$$[-C'_\Gamma, C'_\Gamma] \times [-1, 1] \times [-C'_\Gamma, C'_\Gamma].$$

Because $e^{-d_M(x)} < 1$, the lemma is proven if M is compact.

Now assume M non-compact. Let $\{C_i\}$ be the collection of disjoint cusps of the surface S bounded by a cuspidal horocycles of length $\ell_\Gamma < 1$. By a cusp of M we mean the tangent unit bundle $\tilde{C}_i \subset M$ of a cusp C_i .

By compactness, there exists a constant $K_\Gamma > 0$ such that if $d_M(x) > K_\Gamma$ then the ball of center x and radius 4 is contained in some cusp \tilde{C}_i .

For $d_M(x) \leq K_\Gamma$, the above argument gives a constant $C_\Gamma^{(2)} > 0$ such that α_x is injective on the domain

$$[-C_\Gamma^{(2)} e^{-d_M(x)}, C_\Gamma^{(2)} e^{-d_M(x)}] \times [-1, 1] \times [-C_\Gamma^{(2)} e^{-d_M(x)}, C_\Gamma^{(2)} e^{-d_M(x)}].$$

Let C be any cusp. By conjugating the lattice if necessary, we may assume that the cusp C has a fundamental domain $D = \{z \in \mathbb{H} : |\Re z| \leq 1/2, \text{Im} z > \ell_\Gamma^{-1}\}$ and that a parabolic subgroup Γ stabilizing the cusp C is the subgroup

$$\left\{ \gamma_n := \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} \subset \Gamma.$$

The tangent unit bundle \tilde{D} of D is a fundamental domain of the cusp $\tilde{C} \subset M$.

We consider the usual identification of $\text{SL}(2, \mathbb{R})/\pm I$ with the tangent unit bundle $T^1\mathbb{H}$ the mapping $g \in \text{SL}(2, \mathbb{R})/\pm I \mapsto (g \cdot i, dg_i(i))$. Under this identification, the domain \tilde{D} is identified to

$$\tilde{D} = \left\{ \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} : c^2 + d^2 < \ell_\Gamma, \left| \frac{bd+ac}{c^2+d^2} \right| < \frac{1}{2} \right\}$$

In fact

$$\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot i = \frac{bd+ac}{c^2+d^2} + i \frac{1}{c^2+d^2}.$$

For simplicity, for all $g = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in T^1\mathbb{H} \approx \text{SL}_2(\mathbb{R})/\pm I$, we define $\text{Im}(g) := \text{Im}(g \cdot i)$, i.e. $\text{Im}(g) = (c^2 + d^2)^{-1}$. We also remark that, by the triangle inequality, there exists a constant $c_\Gamma > 0$ such that, if $\bar{x} \in \tilde{D}$ is a representative of $x \in \tilde{C}$, then

$$(148) \quad \log \text{Im}(\bar{x}) - c_\Gamma \leq d_M(x) \leq \log \text{Im}(\bar{x}) + c_\Gamma.$$

Our choice of the constant K_Γ was motivated by the following observation: if $\bar{x} \in \tilde{D}$ and $\bar{x}_1 \in T^1\mathbb{H} \approx \mathrm{SL}_2(\mathbb{R})$ are two representatives of a point $x \in \tilde{C}$ which are at a distance less than 4 from each other, then if $d_M(x) > K_\Gamma$ there exists $n \in \mathbb{Z}$ such that $\bar{x}_1 = \gamma_n \bar{x}$ (in fact $|n| \leq 4 \mathrm{Im} \bar{x}$).

Set, for conciseness,

$$A(t) = \exp(tX/2), \quad H(t) = \exp(tU), \quad \bar{H}(t) = x \exp(tV).$$

Let $I_\Delta = [-\Delta, \Delta] \times [-1, 1] \times [-\Delta, \Delta]$. By continuity, there exist Δ_0 such that for all $\Delta < \Delta_0$ the map $(t, y, z) \in I_\Delta \mapsto H(t)A(y)\bar{H}(z) \in \mathrm{SL}_2(\mathbb{R})$ is a diffeomorphism onto its image satisfying, for all $(t, y, z) \in I_\Delta$,

$$\mathrm{dist}(\mathrm{Id}_{\mathrm{SL}_2(\mathbb{R})}, H(t)A(y)\bar{H}(z)) < 2.$$

Furthermore, by taking partial derivatives of the function

$$F: ((t, y, z), (t', y', z')) \in I_\Delta \times I_\Delta \mapsto H(t)A(y)\bar{H}(z)\bar{H}(z')^{-1}A(y')^{-1}H(t')^{-1}$$

at the point $((0, y, 0), (0, y', 0))$, we find that

$$(149) \quad F((t, y, z), (t', y', z')) = \begin{pmatrix} e^{\frac{y-y'}{2}} & te^{-\frac{y-y'}{2}} - t'e^{-\frac{y-y'}{2}} \\ e^{-\frac{y+y'}{2}}(z-z') & e^{-\frac{y-y'}{2}} \end{pmatrix} + O(\Delta^2).$$

By choosing a smaller Δ_0 , if necessary, we may assume that all the terms $O(\Delta^2)$ appearing in the above identity are bounded by $\Delta/2$.

Let $x \in \tilde{C}$ satisfy $d_M(x) > K_\Gamma$ and let

$$\bar{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$$

be a representative for x belonging to the domain \tilde{D} . We shall show that if $\Delta = \min(\Delta_0, \mathrm{Im}(\bar{x})^{-1}/14)$ the function

$$\alpha_x: (t, y, z) \in I_\Delta \mapsto xH(t)A(y)\bar{H}(z)$$

is injective.

Suppose, by contradiction that this is not the case. Then there exists distinct triplets $(t, y, z) \in I_\Delta$ and $(t', y', z') \in I_\Delta$ such that $\alpha_x(t, y, z) = \alpha_x(t', y', z')$. It follows that the elements $\bar{x}_1 = \bar{x}H(t)A(y)\bar{H}(z)$ and $\bar{x}'_1 = \bar{x}H(t')A(y')\bar{H}(z')$ are distinct representatives of the same point $x_1 \in M$. Since $\mathrm{dist}(\bar{x}, \bar{x}_1) < 2$ and $\mathrm{dist}(\bar{x}, \bar{x}'_1) < 2$ and $\bar{x}_1 \neq \bar{x}'_1$, by a previous observation, there exists $n \in \mathbb{Z} \setminus \{0\}$ such that $\bar{x}_1 = \gamma_n \bar{x}'_1$, that is such that

$$\bar{x}H(t)A(y)\bar{H}(z) = \gamma_n \bar{x}H(t')A(y')\bar{H}(z').$$

Since

$$(150) \quad \bar{x}^{-1} \gamma_n \bar{x} = \begin{pmatrix} 1 + ncd & nd^2 \\ -nc^2 & 1 - ncd \end{pmatrix}$$

the previous identity may be rewritten as

$$\begin{pmatrix} 1 + ncd & nd^2 \\ -nc^2 & 1 - ncd \end{pmatrix} = F((t, y, z), (t', y', z')).$$

From this identity and the identity (149) we obtain

$$\begin{aligned} nd^2 &= te^{-\frac{v-y'}{2}} - t'e^{\frac{v-y'}{2}} + O(\Delta^2) \\ -nc^2 &= e^{-\frac{v+y'}{2}}(z - z') + O(\Delta^2) \end{aligned}$$

and conclude that

$$|n| \operatorname{Im}(\bar{x})^{-1} = |n(c^2 + d^2)| \leq (4e + 2)\Delta < 14\Delta \leq \operatorname{Im}(\bar{x})^{-1}.$$

We proved that $n = 0$, reaching a contradiction. The proof of the Lemma is concluded by observing that by the inequalities (148) the term $\operatorname{Im}(\bar{x})^{-1}$ is equivalent, up to an absolute constant depending only by the lattice, to the term $\exp(-d_M(x))$. \square

Proof of Lemma 6.11. We use the same notation as in the proof of Lemma 6.4. Let $x \in \tilde{C}$, where C is any cusp of the surface S . Let \tilde{D} be the fundamental domain of \tilde{C} described above, and let $\bar{x} \in \tilde{D}$ be a representative of x .

Then by definition of a (β, \mathcal{T}, T) -return, the points $\bar{x}H(t_1)$ and $\bar{x}H(t_0)\bar{H}(z)$ are two representatives for the same point $xH(t_1)$. Then the observation mentioned below (148) gives an integer n such that

$$\bar{x}H(t_1) = \gamma_n \bar{x}H(t_0)\bar{H}(z).$$

Notice that $n \in \mathbb{Z} \setminus \{0\}$, because $z \neq 0$.

Now by (150) and by multiplying on the right by $(H(t_0)H(z))^{-1}$, we get

$$\begin{pmatrix} 1 + ncd & nd^2 \\ -nc^2 & 1 - ncd \end{pmatrix} = \begin{pmatrix} 1 - t_1 z & t_1(1 + t_0 z) - t_0 \\ -z & 1 + t_0 z \end{pmatrix}.$$

The diagonal terms give that $t_1 z = -ncd = t_0 z$. Because $z \neq 0$, it follows that

$$t_0 = t_1.$$

The bottom-left entry now implies that the point $x \exp(t_0 U)$ is a periodic point, with period $z = nc^2$, for the unstable horocycle flow $\{\bar{h}_t\}$. Lemma 6.11 follows from this. \square

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